

# UNIPOTENT REPRESENTATIONS AS A CATEGORICAL CENTRE

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## INTRODUCTION

**0.1.** Let  $\mathbf{k}$  be an algebraic closure of the finite field  $\mathbf{F}_p$  with  $p$  elements. For any power  $q$  of  $p$  let  $\mathbf{F}_q$  be the subfield of  $\mathbf{k}$  with  $q$  elements. Let  $G$  be a reductive connected group over  $\mathbf{k}$ , assumed to be adjoint. Let  $\mathcal{B}$  be the variety of Borel subgroups of  $G$ .

Let  $W$  be the Weyl group of  $G$  and let  $\mathbf{c}$  be a two-sided cell of  $W$ . Let  $s \in \mathbf{Z}_{>0}$  and let  $F : G \rightarrow G$  be the Frobenius map for an  $\mathbf{F}_{p^s}$ -rational structure on  $G$ . Let  $G^F = \{g \in G; F(g) = g\}$ , a finite group. Let  $\text{Rep}^\blacklozenge(G^F)$  (resp.  $\text{Rep}^\mathbf{c}(G^F)$ ) be the category of representations of  $G^F$  over  $\bar{\mathbf{Q}}_l$  which are finite direct sums of unipotent representations in the sense of [DL] (resp. of unipotent representations whose associated two-sided cell (see 1.3) is  $\mathbf{c}$ ); here  $l$  is a fixed prime number invertible in  $\mathbf{k}$ .

In the rest of this subsection we assume for simplicity that the  $\mathbf{F}_{p^s}$ -rational structure on  $G$  is split. The simple objects of  $\text{Rep}^\mathbf{c}(G^F)$  were classified in [L1]. The classification turns out to be the same as that [L4] of unipotent character sheaves on  $G$  whose associated two-sided cell is  $\mathbf{c}$ . The fact that

(a) *these two classification problems have the same solution*  
has not until now been adequately explained.

In [L12] we have shown that the category of perverse sheaves on  $G$  which are direct sums of unipotent character sheaves with associated two-sided cell  $\mathbf{c}$  is naturally equivalent to the centre of a certain monoidal category  $\mathcal{C}^\mathbf{c}\mathcal{B}^2$  of sheaves on  $\mathcal{B}^2$  introduced in [L9] for which the induced ring structure on the Grothendieck group is the  $J$ -ring attached to  $\mathbf{c}$ , see [L10, 18.3]. (The analogous statement for  $D$ -modules on a reductive group over  $\mathbf{C}$  was proved earlier in a quite different way in [BFO].) In this paper we show that  $\text{Rep}^\mathbf{c}(G^F)$  is also naturally equivalent to the centre of  $\mathcal{C}^\mathbf{c}\mathcal{B}^2$  (see 6.3). This implies in particular that the simple objects of  $\text{Rep}^\mathbf{c}(G^F)$  are naturally in bijection with the unipotent character sheaves with associated two-sided cell  $\mathbf{c}$ , which explains (a). It also implies that the set of simple objects  $\text{Rep}_\mathbf{c}(G^F)$  is “independent” of the choice of  $s$ ; in fact, as we show in 7.1,

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it is also independent of the characteristic of  $\mathbf{k}$ . It follows that to classify the unipotent representations of  $G^F$  it is enough to classify the unipotent character sheaves on  $G$  in sufficiently large characteristic; for the latter classification one can use the scheme of [L11] which uses the unipotent support of a character sheaf.

The methods of this paper are extensions of those of [L12]. We replace  $\text{Rep}^c G^F$  by an equivalent category consisting of certain  $G$ -equivariant perverse sheaves on  $G_s$ , the set of all Frobenius maps  $G \rightarrow G$  corresponding to split  $F_{p^s}$ -rational structures on  $G$ ; we view  $G_s$  as an algebraic variety in a natural way. We construct functors  $\underline{\chi}_s, \underline{\zeta}_s$  between this category and the category  $\mathcal{C}^c \mathcal{B}^2$  which are  $q$ -analogues of the truncated induction and truncated restriction  $\underline{\chi}, \underline{\zeta}$  of [L12] and we show that most properties of  $\underline{\chi}, \underline{\zeta}$  are preserved. We also define a truncated convolution product from our sheaves on  $G_s$  and on  $G_{s'}$  to our sheaves on  $G_{s+s'}$  which is analogous to the truncated convolution of character sheaves in [L12]; we also give a meaning for this even when  $s, s'$  are arbitrary integers. The main application of this truncated convolution product is in the case where  $s' = -s$ , the result of the product being a direct sum of character sheaves on  $G$ ; this is used in the proof of a weak form of an adjunction formula between  $\underline{\chi}_s, \underline{\zeta}_s$  which is then used to prove the main result (Theorem 6.3).

**0.2.** In this paper we also prove extensions of the results in 0.1 to the case where  $F : G \rightarrow G$  is the Frobenius map of a nonsplit  $F_{p^s}$ -rational structure. In this case the role of unipotent character sheaves on  $G$  is taken by the unipotent character sheaves on a connected component of the group of automorphism group of  $G$ . Moreover, in this case the centre of  $\mathcal{C}^c \mathcal{B}^2$  is replaced by a slight generalization of the centre (the  $\epsilon$ -centre) which depends on the connected component above.

Many arguments in this paper are very similar to arguments in [L12] and are often replaced by references to the corresponding arguments in [L12].

Our results can be extended to non-unipotent representations and non-unipotent character sheaves; this will be discussed elsewhere. ■

**0.3. Notation.** We assume that we are given a split  $\mathbf{F}_p$ -rational structure on  $G$  with Frobenius map  $F_0 : G \rightarrow G$ . Let  $\nu = \dim \mathcal{B}$ ,  $\Delta = \dim(G)$ ,  $\rho = \text{rk}(G)$ . We shall view  $W$  as an indexing set for the orbits of  $G$  acting on  $\mathcal{B}^2 := \mathcal{B} \times \mathcal{B}$  by simultaneous conjugation; let  $\mathcal{O}_w$  be the orbit corresponding to  $w \in W$  and let  $\bar{\mathcal{O}}_w$  be the closure of  $\mathcal{O}_w$  in  $\mathcal{B}^2$ . For  $w \in W$  we set  $|w| = \dim \mathcal{O}_w - \nu$  (the length of  $w$ ). Let  $w_{\max}$  be the unique element of  $W$  such that  $|w_{\max}| = \nu$ .

As in [L1, 3.1], we say that an automorphism  $\epsilon : W \rightarrow W$  is *ordinary* if it leaves stable the set  $\{s \in W; |s| = 1\}$  and for any two elements  $s \neq s'$  in that set which are in the same orbit of  $\epsilon$ , the product  $ss'$  has order  $\leq 3$ . Let  $\mathfrak{A}$  be the group of ordinary automorphisms of  $W$ .

For  $B \in \mathcal{B}$ , let  $U_B$  be the unipotent radical of  $B$ . Then  $B/U_B$  is independent of  $B$ ; it is “the” maximal torus  $T$  of  $G$ . Let  $\mathcal{X}$  be the group of characters of  $T$ .

Let  $\text{Rep} W$  be the category of finite dimensional representations of  $W$  over  $\mathbf{Q}$ ; let  $\text{Irr} W$  be a set of representatives for the isomorphism classes of irreducible objects

of  $\text{Rep}W$ .

The notation  $\mathcal{D}(X), \mathcal{M}(X), \mathcal{D}_m(X), \mathcal{M}_m(X)$  is as in [L12, 0.2]. (When  $X$  is  $G, \mathcal{B}, \mathcal{O}_w$  or  $\bar{\mathcal{O}}_w$ , the subscript  $m$  refers to the  $\mathbf{F}_{p^{s_0}}$ -structure defined by  $F_0^{s_0}$  for a sufficiently large  $s_0 > 0$ .) For  $K \in \mathcal{D}(X)$ ,  $\mathcal{H}^i K, \mathcal{H}_x^i K, K^i, K[[m]] = K[n](n/2)$ ,  $\mathfrak{D}(K)$  are as in [L12, 0.2]. For  $K \in \mathcal{M}_m(X)$ ,  $gr_j K$  is as in [L12, 0.2]. For  $K \in \mathcal{D}_m(X)$ ,  $K^{\{i\}} = gr_i(K^i)(i/2)$ , is as in [L12, 0.2].

If  $K \in \mathcal{M}(X)$  and  $A$  is a simple object of  $\mathcal{M}(X)$  we denote by  $(A : K)$  the multiplicity of  $A$  in a Jordan-Hölder series of  $K$ . The notation  $C \simeq \{C_i; i \in I\}$  is as in [L12, 0.2].

If  $X, X'$  are algebraic varieties over  $\mathbf{k}$ , we say that a map of sets  $f : X \rightarrow X'$  is a quasi-morphism if for some  $\mathbf{F}_q$ -rational structure on  $X$  and  $X'$  with Frobenius maps  $F$  and  $F'$  and some integer  $t \geq 0$ ,  $fF^t : X \rightarrow X'$  is a morphism equal to  $F'^t f$ . If, in addition,  $fF = F't$  then we have well defined functors  $f_! : \mathcal{D}_m(X) \rightarrow \mathcal{D}_m(X')$ ,  $f^* : \mathcal{D}_m(X') \rightarrow \mathcal{D}_m(X)$  such that  $f_!$  is the composition of usual functors  $(fF^t)_!(F^t)^* = (F'^t)^*(F'^t f)_!$  and  $f^*$  is the composition of usual functors  $(F^t)_!(fF^t)^* = (F'^t f)^*(F'^t)_!$ . The usual properties of  $f_!, f^*$  for morphisms continue to hold for quasi-morphisms.

We will denote by  $\mathbf{p}$  the variety consisting of one point. For any variety  $X$  let  $\mathfrak{L}_X = \alpha_! \bar{\mathbf{Q}}_l \in \mathcal{D}_m X$  where  $\alpha : X \times T \rightarrow X$  is the obvious projection. We sometimes write  $\mathfrak{L}$  instead of  $\mathfrak{L}_X$ .

Let  $v$  be an indeterminate. For any  $\phi \in \mathbf{Q}[v, v^{-1}]$  and any  $k \in \mathbf{Z}$  we write  $(k; \phi)$  for the coefficient of  $v^k$  in  $\phi$ . Let  $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ .

## CONTENTS

1. Truncated induction.
2. Truncated restriction.
3. Truncated convolution from  $G_{\epsilon, s} \times G_{\epsilon', s'}$  to  $G_{\epsilon\epsilon', s+s'}$ .
4. Analysis of the composition  $\underline{\zeta}_{\epsilon, s} \underline{\chi}_{\epsilon, s}$ .
5. Adjunction formula (weak form).
6. Equivalence of  $\mathcal{C}^c G_{\epsilon, s}$  with the  $\epsilon$ -centre of  $\mathcal{C}^c \mathcal{B}^2$ .
7. Relation with Soergel bimodules.

### 1. TRUNCATED INDUCTION

**1.1.** For  $y \in W$  let  $L_y \in \mathcal{D}_m(\mathcal{B}^2)$  be the constructible sheaf which is  $\bar{\mathbf{Q}}_l$  (with the standard mixed structure of pure weight 0) on  $\mathcal{O}_y$  and is 0 on  $\mathcal{B}^2 - \mathcal{O}_y$ ; let  $L_y^\sharp \in \mathcal{D}_m(\mathcal{B}^2)$  be its extension to an intersection cohomology complex of  $\bar{\mathcal{O}}_y$  (equal to 0 on  $\mathcal{B}^2 - \bar{\mathcal{O}}_y$ ). Let  $\mathbf{L}_y = L_y^\sharp[[|y| + \nu]] \in \mathcal{D}_m(\mathcal{B}^2)$ .

Let  $r \geq 1$ . For  $\mathbf{w} = (w_1, w_2, \dots, w_r) \in W^r$  we set  $|\mathbf{w}| = |w_1| + \dots + |w_r|$ . Let  $L_{\mathbf{w}}^{[1, r]} \in \mathcal{D}_m(\mathcal{B}^{r+1})$  be as in [L12, 1.1]. For any  $J \subset [1, r]$  let  $L_{\mathbf{w}}^J \in \mathcal{D}_m(\mathcal{B}^{r+1})$ ,  $\dot{L}_{\mathbf{w}}^J \in \mathcal{D}_m(\mathcal{B}^{r+1})$  be as in [L12, 1.1]. As in [L12, 1.1(a)], we have a distinguished triangle

$$(a) \quad (L_{\mathbf{w}}^J, L_{\mathbf{w}}^{[1, r]}, \dot{L}_{\mathbf{w}}^J)$$

in  $\mathcal{D}_m(\mathcal{B}^{r+1})$ . For any  $i < i'$  in  $[1, r]$  let  $p_{i,i'} : \mathcal{B}^{r+1} \rightarrow \mathcal{B}^2$  be the projection to the  $i, i'$  factors. For  ${}^1L, {}^2L, \dots, {}^rL$  in  $\mathcal{D}_m(\mathcal{B}^2)$  we set

$${}^1L \bullet {}^2L \bullet \dots \bullet {}^rL = p_{0r!}(p_{01}^* {}^1L \otimes p_{12}^* {}^2L \otimes \dots \otimes p_{r-1,r}^* {}^rL) \in \mathcal{D}_m(\mathcal{B}^2).$$

**1.2.** Let  $\mathbf{H}$  be the free  $\mathcal{A}$ -module with basis  $\{T_w; w \in W\}$ . It is well known that  $\mathbf{H}$  has a unique structure of associative  $\mathcal{A}$ -algebra with  $1 = T_1$  (Hecke algebra) such that  $T_w T_{w'} = T_{ww'}$  if  $w, w' \in W$ ,  $|ww'| = |w| + |w'|$  and  $T_s^2 = 1 + (v - v^{-1})T_s$  if  $s \in W$ ,  $|s| = 1$ . Let  $\{c_w; w \in W\}$  be the “new” basis of  $\mathbf{H}$  defined as in [L10, 5.2] with  $L(w) = |w|$ .

For  $x, y \in W$ , the relations  $x \preceq y$ ,  $x \sim y$ ,  $x \sim_L y$  on  $W$  are defined as in [L12, 1.3]. If  $\mathbf{c}$  is a two-sided cell of  $W$  and  $w \in W$ , the relations  $w \preceq \mathbf{c}$ ,  $\mathbf{c} \preceq w$ ,  $w \prec \mathbf{c}$ ,  $\mathbf{c} \prec w$  are defined as in [L12, 1.3]. If  $\mathbf{c}, \mathbf{c}'$  are two-sided cells of  $W$ , the relations  $\mathbf{c} \preceq \mathbf{c}'$ ,  $\mathbf{c} \prec \mathbf{c}'$  are defined as in [L12, 1.3]. Let  $\mathbf{a} : W \rightarrow \mathbf{N}$  be the  $\mathbf{a}$ -function in [L10, 13.6]. If  $\mathbf{c}$  is a two-sided cell of  $W$ , then for all  $w \in \mathbf{c}$  we have  $\mathbf{a}(w) = \mathbf{a}(\mathbf{c})$  where  $\mathbf{a}(\mathbf{c})$  is a constant.

Let  $\mathbf{J}$  be the free  $\mathbf{Z}$ -module with basis  $\{t_z; z \in W\}$  with the structure of associative ring (with 1) as in [L12, 1.3]. For a two-sided cell  $\mathbf{c}$  of  $W$  let  $\mathbf{J}^{\mathbf{c}}$  be the subgroup of  $\mathbf{J}$  generated by  $\{t_z; z \in \mathbf{c}\}$ ; it is a subring of  $\mathbf{J}$  with unit element  $\sum_{d \in \mathbf{D}_{\mathbf{c}}} t_d$  where  $\mathbf{D}_{\mathbf{c}}$  is the set of distinguished involutions of  $\mathbf{c}$ . We have  $\mathbf{J} = \bigoplus_{\mathbf{c}} \mathbf{J}^{\mathbf{c}}$  as rings.

For  $E \in \text{Irr} W$  we define a simple  $\mathbf{Q} \otimes \mathbf{J}$ -module  $E_{\infty}$  and a simple  $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathbf{H}$ -module  $E(v)$  as in [L12, 1.3]; there is a unique two-sided cell  $\mathbf{c}_E$  of  $W$  such that  $\mathbf{J}^{\mathbf{c}_E} E_{\infty} \neq 0$ .

Let  $\epsilon \in fA$ . Let  $E \in \text{Irr} W$ . We say that  $E \in \text{Irr}_{\epsilon} W$  if  $\text{tr}(\epsilon(w), E) = \text{tr}(w, E)$  for any  $w \in W$ . In this case there exists a linear transformation of finite order  $\epsilon_E : E \rightarrow E$  such that  $\epsilon_E w \epsilon_E^{-1} = \epsilon(w) : E \rightarrow E$  for any  $w \in W$ ; moreover  $\epsilon_E$  is unique up to multiplication by  $-1$ . See [L1, 3.2]). For each  $E \in \text{Irr}_{\epsilon} W$  we choose  $\epsilon_E$  as above. As a  $\mathbf{Q}$ -vector space we have  $E_{\infty} = E$ ,  $E(v) = \mathbf{Q}(v) \otimes_{\mathbf{Q}} E$  hence, if  $E \in \text{Irr}_{\epsilon} W$ ,  $\tilde{e} : E \rightarrow E$  can be viewed as a  $\mathbf{Q}$ -linear map (of finite order)  $\tilde{e} : E_{\infty} \rightarrow E_{\infty}$  and as a  $\mathbf{Q}(v)$ -linear map (of finite order)  $\tilde{e} : E(v) \rightarrow E(v)$ . From the definitions we see that  $\tilde{e} t_w \tilde{e}^{-1} = t_{\epsilon(w)} : E_{\infty} \rightarrow E_{\infty}$  and  $\tilde{e} T_w \tilde{e}^{-1} = T_{\epsilon(w)} : E(v) \rightarrow E(v)$  for any  $w \in W$ .

If  $E \in \text{Irr}_{\epsilon} W$  then  $\epsilon(\mathbf{c}_E) = \mathbf{c}_E$ . Let  $\text{Irr}_{\epsilon, \mathbf{c}} W = \{E \in \text{Irr}_{\epsilon} W; \mathbf{c}_E = \mathbf{c}\}$ .

**1.3.** For any  $\epsilon \in \mathfrak{A}$ ,  $s \in \mathbf{Z}$  let  $G_{\epsilon, s}$  be the set of bijections  $F : G \rightarrow G$  such that

- (i) if  $s > 0$  then  $F$  is the Frobenius map for an  $F_{p^s}$ -rational structure on  $G$ ;
- (ii) if  $s < 0$  then  $F^{-1}$  is the Frobenius map for an  $F_{p^{-s}}$ -rational structure on  $G$ ;
- (iii) if  $s = 0$  then  $F$  is an automorphism of  $G$ ;

moreover in each case (i)–(iii) we require that the following holds: for any  $w \in W$  and any  $(B, B') \in \mathcal{O}_w$  we have  $(F(B), F(B')) \in \mathcal{O}_{\epsilon(w)}$ .

(If  $\epsilon = 1, s = 0$  we can identify  $G$  and  $G_{\epsilon, s}$  by  $g \mapsto \text{Ad}(g)$ .) Now  $G$  acts on  $G_{\epsilon, s}$  by  $g : F \mapsto \text{Ad}(g) F \text{Ad}(g^{-1})$ . If  $s \neq 0$ , this action is transitive and the stabilizer of a

point  $F \in G_{\epsilon,s}$  is the finite group  $G^F = \{g \in G; F(g) = g\}$ . For any  $s \in \mathbf{Z}$  and any  $\tilde{F} \in G_{\epsilon,s}$ , the maps  $\lambda : G \rightarrow G_{\epsilon,s}$ ,  $g \mapsto \text{Ad}(g)\tilde{F}$  and  $\lambda' : G \rightarrow G_{\epsilon,s}$ ,  $g \mapsto \tilde{F}\text{Ad}(g)$  are bijections (by Lang's theorem); we use  $\lambda$  (resp.  $\lambda'$ ) to view  $G_{\epsilon,s}$  with  $s \geq 0$  (resp.  $s \leq 0$ ) as an affine algebraic variety isomorphic to  $G$ ; this algebraic variety structure on  $G_{\epsilon,s}$  is independent of the choice of  $\tilde{F}$ . We have  $\dim G_{\epsilon,s} = \Delta$ . The  $G$ -action above on  $G_{\epsilon,s}$  is an algebraic group action. When  $X = G_{\epsilon,s}$  then the subscript  $m$  in  $\mathcal{D}_m(X), \mathcal{M}_m(X)$  refers to the  $\mathbf{F}_{p^{s_0}}$ -structure with Frobenius map  $F \mapsto F_0^{s_0} F F_0^{-s_0}$  (with  $F_0, s_0$  as in 0.3).

Note that  $\sqcup_{\epsilon \in \mathfrak{A}, s \in \mathbf{Z}} G_{\epsilon,s}$  is a group under composition of maps: if  $F \in G_{\epsilon,s}, F' \in G_{\epsilon',s'}$  then  $FF' \in G_{\epsilon\epsilon',s+s'}$ . (It is enough to show that for some  $F \in G_{\epsilon,s}, F' \in G_{\epsilon',s'}$  we have  $FF' \in G_{\epsilon\epsilon',s+s'}$ . We take  $F = \text{Ad}(\gamma)F_0^s, F' = \text{Ad}(\gamma')F_0^{s'}$  where  $\gamma \in G_{\epsilon,1}$  and  $\gamma' \in G_{\epsilon',1}$  commute with  $F_0$ ; then  $FF' = \text{Ad}(\gamma\gamma')F_0^{s+s'}$  and  $\gamma\gamma' \in G_{\epsilon\epsilon',1}$  commutes with  $F_0$  hence  $FF' \in G_{\epsilon\epsilon',s+s'}$ .) Note that the composition  $G_{\epsilon,s} \times G_{\epsilon',s'} \rightarrow G_{\epsilon\epsilon',s+s'}$  is not in general a morphism of algebraic varieties but only a quasi-morphism (see 0.3), which is good enough for our purposes.

*Until the end of Section 2 we fix  $\epsilon \in \mathfrak{A}$ .*

Let  $s \in \mathbf{Z}$ . We consider the maps  $\mathcal{B}^2 \xleftarrow{f} X_{\epsilon,s} \xrightarrow{\pi} G_{\epsilon,s}$  where

$$\begin{aligned} X_{\epsilon,s} &= \{(B, B', F) \in \mathcal{B} \times \mathcal{B} \times G_{\epsilon,s}; F(B) = B'\}, \\ f(B, B', F) &= (B, B'), \pi(B, B', F) = F. \end{aligned}$$

Now  $L \mapsto \chi_{\epsilon,s}(L) = \pi_! f^* L$  defines a functor  $\mathcal{D}_m(\mathcal{B}^2) \rightarrow \mathcal{D}_m(G_{\epsilon,s})$ . (When  $\epsilon = 1, s = 0$ ,  $\chi_{\epsilon,s}$  coincides with the functor  $\chi$  defined in [L12, 1.5]). For  $i \in \mathbf{Z}, L \in \mathcal{D}_m(\mathcal{B}^2)$  we write  $\chi_{\epsilon,s}^i(L)$  instead of  $(\chi_{\epsilon,s}(L))^i$ . For any  $z \in W$  we set  $R_{\epsilon,s,z} = \chi_{\epsilon,s}(L_z^\sharp) \in \mathcal{D}_m(G_{\epsilon,s})$ . (When  $\epsilon = 1, s = 0$  this is the same as  $R_z$  in [L12, 1.5].)

Let  $b : G_{\epsilon^{-1},-s} \xrightarrow{\sim} G_{\epsilon,s}$  be the isomorphism  $F \mapsto F^{-1}$  and let  $b' : \mathcal{B}^2 \xrightarrow{\sim} \mathcal{B}^2$  be the isomorphism  $(B, B') \mapsto (B', B)$ . From the definitions we see that for  $L \in \mathcal{D}_m(\mathcal{B}^2)$  we have  $\chi_{\epsilon,s}(b'_! L) = b_! \chi_{\epsilon^{-1},-s}(L)$ .

Let  $CS(G_{\epsilon,s})$  be a set of representatives for the isomorphism classes of simple perverse sheaves  $A \in \mathcal{M}(G_{\epsilon,s})$  such that  $(A : R_{\epsilon,s,z}^j) \neq 0$  for some  $z \in W, j \in \mathbf{Z}$ . (When  $\epsilon = 1, s = 0$  this agrees with the definition of  $CS(G)$  in [L12, 1.5].) Now let  $A \in CS(G_{\epsilon,s})$ . We associate to  $A$  a two-sided cell  $\mathbf{c}_A$  as follows.

Assume first that  $s \neq 0$ . Since  $A$  is  $G$ -equivariant and the conjugation action of  $G$  on  $G_{\epsilon,s}$  is transitive, for any  $F \in G_{\epsilon,s}$  we have  $A|_{\{F\}} = r_{A,F}[\Delta]$  where  $r_{A,F}$  is an irreducible  $G^F$ -module. From the definitions, for any  $z \in W$  and any  $F \in G_{\epsilon,s}$  we have

$$(A : R_{s,z}^j) = (r_{A,F} : IH^{j-\Delta}\{(B; (B, FB) \in \bar{\mathcal{O}}_z)\})_{G^F}$$

where the right hand side is the multiplicity of  $r_{A,F}$  in the  $G^F$ -module

$$IH^{j-\Delta}\{(B; (B, FB) \in \bar{\mathcal{O}}_z)\};$$

here  $IH$  denotes intersection cohomology with coefficients in  $\bar{\mathbf{Q}}_l$ . In particular,  $r_{A,F}$  is a unipotent representation of  $G^F$ . By [L1, 3.8], for any  $A \in CS(G_{\epsilon,s})$ , any  $F \in G_{\epsilon,s}$ , any  $z \in W$  and any  $j \in \mathbf{Z}$  we have

$$\begin{aligned} (r_{A,F} : IH^{j-\Delta} \{(B; (B, FB) \in \bar{O}_z)\}_{G^F}) \\ = (j - \Delta - |z|; (-1)^{j-\Delta} \sum_{E \in \text{Irr}_\epsilon W} c_{A,E,\bar{e}} \text{tr}(\tilde{e}c_z, E(v))) \end{aligned}$$

or equivalently

$$(a) \quad (A : R_{\epsilon,s,z}^j) = (j - \Delta - |z|; (-1)^{j-\Delta} \sum_{E \in \text{Irr}_\epsilon W} c_{A,E,\bar{e}} \text{tr}(\tilde{e}c_z, E(v)))$$

where  $c_{A,E,\epsilon}$  are uniquely defined rational numbers; now (a) also holds when  $s = 0$ , see [L12, 1.5(a)] when  $\epsilon = 1$  and [L6, 34.19, 35.22], [L8, 44.7(e)] for general  $\epsilon$ . Moreover, if  $s \neq 0$  then, by [L1, 6.17], given  $A$  as above, there is a unique two-sided cell  $\mathbf{c}_A$  of  $W$  such that  $\epsilon(\mathbf{c}_A) = \mathbf{c}_A$  and  $c_{A,E,\epsilon} = 0$  whenever  $E \in \text{Irr}_\epsilon W$  satisfies  $\mathbf{c}_E \neq \mathbf{c}_A$ . The same holds when  $s = 0$ , see [L12, 1.5] when  $\epsilon = 1$  and [L7, §41] for general  $\epsilon$ .

When  $s \neq 0$ ,  $\mathbf{c}_A$  differs from the two-sided cell associated to  $r_{A,F}$  in [L1, 4.23] by multiplication on the left or right by  $w_{\max}$ . Similarly, when  $s = 0$ ,  $\mathbf{c}_A$  differs from the two-sided cell associated to  $A$  in [L7, §41] by multiplication on the left or right by  $w_{\max}$ .

As in [L12, 1.5(b)], for  $s \in \mathbf{Z}$  we have

(b)  $(A : R_{\epsilon,s,z}^j) \neq 0$  for some  $z \in \mathbf{c}_A, j \in \mathbf{Z}$  and conversely, if  $(A : R_{\epsilon,s,z}^j) \neq 0$  for  $z \in W, j \in \mathbf{Z}$ , then  $\mathbf{c}_A \preceq z$ .

For  $s \in \mathbf{Z}$ ,  $A \in CS(G_{\epsilon,s})$  let  $a_A$  be the value of the  $\mathbf{a}$ -function on  $\mathbf{c}_A$ . If  $z \in W, E \in \text{Irr}_\epsilon W$  satisfy  $\text{tr}(\tilde{e}c_z, E(v)) \neq 0$  then  $\mathbf{c}_E \preceq z$ ; if in addition we have  $z \in \mathbf{c}_E$ , then

$$\text{tr}(\tilde{e}c_z, E(v)) = \gamma_{z,E,\bar{e}} v^{a_E} + \text{lower powers of } v$$

where  $\gamma_{z,E,\epsilon} \in \mathbf{Z}$  and  $a_E$  is the value of the  $\mathbf{a}$ -function on  $\mathbf{c}_E$ . Hence from (a) we see that

(c)  $(A : R_{\epsilon,s,z}^j) = 0$  unless  $\mathbf{c}_A \preceq z$  and, if  $z \in \mathbf{c}_A$ , then

$$\begin{aligned} (A : R_{\epsilon,s,z}^j) \\ = (-1)^{j+\Delta} (j - \Delta - |z|; (\sum_{E \in \text{Irr}_\epsilon W; \mathbf{c}_E = \mathbf{c}_A} c_{A,E,\bar{e}} \gamma_{z,E,\bar{e}}) v^{a_A} + \text{lower powers of } v)) \end{aligned}$$

which is 0 unless  $j - \Delta - |z| \leq a_A$ .

In the remainder of this section we fix a two-sided cell  $\mathbf{c}$  of  $W$  such that  $\epsilon(\mathbf{c}) = \mathbf{c}$ ; we set  $a = \mathbf{a}(\mathbf{c})$ .

For  $s \in \mathbf{Z}$  and  $Y = G_{\epsilon,s}$  or  $Y = \mathcal{B}^2$  let  $\mathcal{M}^\spadesuit Y$  be the category of perverse sheaves

on  $Y$  whose composition factors are all of the form  $A \in CS(G_{\epsilon,s})$ , when  $Y = G_{\epsilon,s}$ , or of the form  $\mathbf{L}_z$  with  $z \in W$ , when  $Y = \mathcal{B}^2$ . Let  $\mathcal{M}^{\preceq}Y$  (resp.  $\mathcal{M}^{\prec}Y$ ) be the category of perverse sheaves on  $Y$  whose composition factors are all of the form  $A \in CS(G_{\epsilon,s})$  with  $\mathbf{c}_A \preceq \mathbf{c}$  (resp.  $\mathbf{c}_A \prec \mathbf{c}$ ), when  $Y = G_{\epsilon,s}$ , or of the form  $\mathbf{L}_z$  with  $z \preceq \mathbf{c}$  (resp.  $z \prec \mathbf{c}$ ) when  $Y = \mathcal{B}^2$ . Let  $\mathcal{D}^{\spadesuit}Y$  (resp.  $\mathcal{D}^{\preceq}Y$  or  $\mathcal{D}^{\prec}Y$ ) be the category of all  $K \in \mathcal{D}(Y)$  such that  $K^i \in \mathcal{M}^{\spadesuit}Y$  (resp.  $K^i \in \mathcal{M}^{\preceq}Y$  or  $K^i \in \mathcal{M}^{\prec}Y$ ) for all  $i \in \mathbf{Z}$ . Let  $\mathcal{M}_m^{\spadesuit}Y$  (or  $\mathcal{M}_m^{\preceq}Y$ , or  $\mathcal{M}_m^{\prec}Y$ ) be the category of all  $K \in \mathcal{M}_mY$  which are also in  $\mathcal{M}^{\spadesuit}Y$  (or  $\mathcal{M}^{\preceq}Y$  or  $\mathcal{M}^{\prec}Y$ ). Let  $\mathcal{D}_m^{\spadesuit}Y$  (or  $\mathcal{D}_m^{\preceq}Y$ , or  $\mathcal{D}_m^{\prec}Y$ ) be the category of all  $K \in \mathcal{D}_mY$  which are also in  $\mathcal{D}^{\spadesuit}Y$  (or  $\mathcal{D}^{\preceq}Y$  or  $\mathcal{D}^{\prec}Y$ ). From (c) we deduce:

(d) *If  $z \preceq \mathbf{c}$ , then  $R_{\epsilon,s,z}^j \in \mathcal{M}^{\preceq}G_{\epsilon,s}$  for all  $j \in \mathbf{Z}$ . If  $z \in \mathbf{c}$  and  $j > a + \Delta + |z|$ , then  $R_{\epsilon,s,z}^j \in \mathcal{M}^{\prec}G_{\epsilon,s}$ . If  $z \prec \mathbf{c}$  then  $R_{\epsilon,s,z}^j \in \mathcal{M}^{\prec}G_{\epsilon,s}$  for all  $j \in \mathbf{Z}$ .*

**Lemma 1.4.** *Let  $s \in \mathbf{Z}$ . Let  $r \geq 1$ ,  $J \subset [1, r]$ ,  $J \neq \emptyset$  and  $\mathbf{w} = (w_1, w_2, \dots, w_r) \in W^r$ . Let  $\mathfrak{E} = \Delta + ra$ .*

(a) *Assume that  $w_i \in \mathbf{c}$  for some  $i \in [1, r]$ . If  $j \in \mathbf{Z}$  (resp.  $j > \mathfrak{E}$ ) then  $\chi_{\epsilon,s}^j(p_{0r}!L_{\mathbf{w}}^{[1,r]}[|\mathbf{w}|])$  is in  $\mathcal{M}^{\preceq}G_{\epsilon,s}$  (resp.  $\mathcal{M}^{\prec}G_{\epsilon,s}$ ).*

(b) *Assume that  $w_i \in \mathbf{c}$  for some  $i \in J$ . If  $j \in \mathbf{Z}$  (resp.  $j \geq \mathfrak{E}$ ) then  $\chi_{\epsilon,s}^j(p_{0r}!\dot{L}_{\mathbf{w}}^J[|\mathbf{w}|])$  is in  $\mathcal{M}^{\preceq}G_{\epsilon,s}$  (resp.  $\mathcal{M}^{\prec}G_{\epsilon,s}$ ).*

(c) *Assume that  $w_i \in \mathbf{c}$  for some  $i \in J$ . If  $j \geq \mathfrak{E}$  then the cokernel of the map*

$$\chi_{\epsilon,s}^j(p_{0r}!L_{\mathbf{w}}^J[|\mathbf{w}|]) \rightarrow \chi_{\epsilon,s}^j(p_{0r}!L_{\mathbf{w}}^{[1,r]}[|\mathbf{w}|])$$

*associated to 1.1(a) is in  $\mathcal{M}^{\prec}G_{\epsilon,s}$ .*

(d) *Assume that  $w_i \in \mathbf{c}$  for some  $i \in J$ . If  $j \in \mathbf{Z}$  (resp.  $j > \mathfrak{E}$ ) then  $\chi_{\epsilon,s}^j(p_{0r}!L_{\mathbf{w}}^J[|\mathbf{w}|])$  is in  $\mathcal{M}^{\preceq}G_{\epsilon,s}$  (resp.  $\mathcal{M}^{\prec}G_{\epsilon,s}$ ).*

(e) *Assume that  $w_i \prec \mathbf{c}$  for some  $i \in J$ . If  $j \in \mathbf{Z}$  then  $\chi_{\epsilon,s}^j(p_{0r}!L_{\mathbf{w}}^{[1,r]}[|\mathbf{w}|]) \in \mathcal{M}^{\prec}G_{\epsilon,s}$  and  $\chi_{\epsilon,s}^j(p_{0r}!L_{\mathbf{w}}^J[|\mathbf{w}|]) \in \mathcal{M}^{\prec}G_{\epsilon,s}$ .*

When  $\epsilon = 1, s = 0$  this is just [L12, 1.6]; the proof in the general case is entirely similar (it uses 1.3(b), 1.3(c)).

**1.5.** Let  $s \in \mathbf{Z}$ . Let  $CS_{\epsilon,s,\mathbf{c}} = \{A \in CS(G_{\epsilon,s}); \mathbf{c}_A = \mathbf{c}\}$ . For any  $z \in \mathbf{c}$  we set  $n_z = a + \Delta + |z|$ . Let  $A \in CS_{\epsilon,s,\mathbf{c}}$  and let  $z \in \mathbf{c}$ . We have:

$$(a) \quad (A : R_{s,z}^{n_z}) = (-1)^{a+|z|} \sum_{E \in \text{Irr}_{\epsilon,\mathbf{c}}W} c_{A,E,\tilde{e}} \text{tr}(\tilde{e}t_z, E_{\infty}).$$

When  $\epsilon = 1, s = 0$  this is just [L12, 1.7(a)]. In the general case, from 1.3(a) we have

$$(A : R_{s,z}^{n_z}) = (-1)^{a+|z|} \sum_{E \in \text{Irr}_{\epsilon}W} c_{A,E,\tilde{e}}(a; \text{tr}(\tilde{e}c_z, E(v)))$$

and it remains to use that  $(a; \text{tr}(\tilde{e}c_z, E(v)))$  is equal to  $\text{tr}(\tilde{e}t_z, E_{\infty})$  if  $E \in \text{Irr}_{\epsilon,\mathbf{c}}W$  and to 0, otherwise. We have:

(b) *For any  $A \in CS_{\epsilon,s,\mathbf{c}}$  there exists  $z \in \mathbf{c}$  such that  $(A : R_{\epsilon,s,z}^{n_z}) \neq 0$ .*

The proof, based on (a), is the same as that in the case  $\epsilon = 1, s = 0$  given in [L12, 1.7(b)].

Let  $\mathbf{c}^0 = \{z \in \mathbf{c}; z \sim_L \epsilon(z^{-1})\}$ . If  $z \in \mathbf{c} - \mathbf{c}^0$  and  $E \in \text{Irr}_{\epsilon, \mathbf{c}} W$ , then  $\text{tr}(\tilde{e}t_z, E_\infty) = 0$ . (We can write  $E_\infty = \bigoplus_{d \in \mathbf{D}_\epsilon} t_d E_\infty$  and  $\tilde{e}t_z : E_\infty \rightarrow E_\infty$  maps the summand  $t_d E_\infty$  (where  $z \sim_L d$ ) into  $t_{\epsilon(d')} E_\infty$  (where  $d' \in \mathbf{D}_\epsilon$ ,  $d' \sim_L z^{-1}$ ) and all other summands to 0. If  $\text{tr}(\tilde{e}t_z, E_\infty) \neq 0$ , we must have  $t_d E_\infty = t_{\epsilon(d')} E_\infty \neq 0$  and  $d = \epsilon(d')$  and  $z \sim_L \epsilon(z^{-1})$ .) From this and (a) we deduce

(c) *If  $z \in \mathbf{c} - \mathbf{c}^0$ , then  $R_{\epsilon, s, z}^{n_z} = 0$ .*

**1.6.** Let  $s \in \mathbf{Z}$ . For  $Y = G_{\epsilon, s}$  or  $\mathcal{B}^2$  let  $\mathcal{C}^\spadesuit Y$  be the subcategory of  $\mathcal{M}^\spadesuit Y$  consisting of semisimple objects; let  $\mathcal{C}_0^\spadesuit Y$  be the subcategory of  $\mathcal{M}_m Y$  consisting of those  $K \in \mathcal{M}_m Y$  such that  $K$  is pure of weight 0 and such that as an object of  $\mathcal{M}(Y)$ ,  $K$  belongs to  $\mathcal{C}^\spadesuit Y$ . Let  $\mathcal{C}^c Y$  be the subcategory of  $\mathcal{M}^\spadesuit Y$  consisting of objects which are direct sums of objects in  $CS_{\epsilon, s, \mathbf{c}}$  (if  $Y = G_{\epsilon, s}$ ) or of the form  $\mathbf{L}_z$  with  $z \in \mathbf{c}$  (if  $Y = \mathcal{B}^2$ ). Let  $\mathcal{C}_0^c Y$  be the subcategory of  $\mathcal{C}_0^\spadesuit Y$  consisting of those  $K \in \mathcal{C}_0^\spadesuit Y$  such that as an object of  $\mathcal{C}^\spadesuit Y$ ,  $K$  belongs to  $\mathcal{C}^c Y$ . For  $K \in \mathcal{C}_0^\spadesuit Y$ , let  $\underline{K}$  be the largest subobject of  $K$  such that, as an object of  $\mathcal{C}^\spadesuit Y$ , we have  $\underline{K} \in \mathcal{C}^c Y$ .

For  $L \in \mathcal{C}_0^\spadesuit \mathcal{B}^2$  we define  ${}^\epsilon L \in \mathcal{C}_0^\spadesuit \mathcal{B}^2$  as follows. We have canonically  $L = \bigoplus_{y \in W} V_y \otimes \mathbf{L}_y$  where  $V_y$  are finite dimensional  $\bar{\mathbf{Q}}_l$ -vector spaces; we set  ${}^\epsilon L = \bigoplus_{y \in W} V_y \otimes \mathbf{L}_{\epsilon^{-1}(y)}$ . We show:

(a) *Let  $s \in \mathbf{N}$ . Define  $u : G_{\epsilon, s} \times \mathcal{B}^2 \rightarrow G_{\epsilon, s} \times \mathcal{B}^2$  by*

$$(F, (B_1, B_2)) \mapsto (F, F(B_1), F(B_2))$$

*and let  $L \in \mathcal{C}_0^\spadesuit \mathcal{B}^2$ . We have canonically  $u^*(\bar{\mathbf{Q}}_l \boxtimes L) = \bar{\mathbf{Q}}_l \boxtimes {}^\epsilon L$ .*

We can assume that  $L = \mathbf{L}_y$  where  $y \in W$ ; we must show that  $u^*(\bar{\mathbf{Q}}_l \boxtimes \mathbf{L}_y) = \bar{\mathbf{Q}}_l \boxtimes \mathbf{L}_{\epsilon^{-1}(y)}$  or that  $u^*(\bar{\mathbf{Q}}_l \boxtimes L_y^\sharp) = \bar{\mathbf{Q}}_l \boxtimes L_{\epsilon^{-1}(y)}^\sharp$ . Now  $\bar{\mathbf{Q}}_l \boxtimes L_y^\sharp$  is the intersection cohomology complex of  $G_{\epsilon, s} \times \bar{\mathcal{O}}_y$  with coefficients in  $\bar{\mathbf{Q}}_l$  (extended by 0 on  $G_{\epsilon, s} \times (\mathcal{B}^2 - \bar{\mathcal{O}}_y)$ ). Hence  $u^*(\bar{\mathbf{Q}}_l \boxtimes L_y^\sharp)$  is the intersection cohomology complex of  $u^{-1}(G_{\epsilon, s} \times \bar{\mathcal{O}}_y)$  with coefficients in  $\bar{\mathbf{Q}}_l$  (extended by 0 on  $G_{\epsilon, s} \times u^{-1}(\mathcal{B}^2 - \bar{\mathcal{O}}_y)$ ) that is, the intersection cohomology complex of  $G_{\epsilon, s} \times \bar{\mathcal{O}}_{\epsilon^{-1}(y)}$  with coefficients in  $\bar{\mathbf{Q}}_l$  (extended by 0 on  $G_{\epsilon, s} \times (\mathcal{B}^2 - \bar{\mathcal{O}}_{\epsilon^{-1}(y)})$ ). This is  $\bar{\mathbf{Q}}_l \boxtimes L_{\epsilon^{-1}(y)}^\sharp$ , as required.

Assume that  $s \in \mathbf{Z}_{>0}$  and let  $F \in G_{\epsilon, s}$ . For any  $A \in \mathcal{C}^\spadesuit G_{\epsilon, s}$  we have  $A|_{\{F\}} = r_{A, F}[\Delta]$  where  $r_{A, F} \in \text{Rep}^\spadesuit(G^F)$  (see 0.1). Moreover, from the definitions we see that

(b)  *$A \mapsto r_{A, F}$  is an equivalence of categories  $\mathcal{C}^c G_{\epsilon, s} \xrightarrow{\sim} \text{Rep}^c(G^F)$  (see 0.1).*

**Proposition 1.7.** *Let  $s \in \mathbf{Z}$ .*

- (a) *If  $L \in \mathcal{D}^\preceq \mathcal{B}^2$  then  $\chi_{\epsilon, s}(L) \in \mathcal{D}^\preceq G_{\epsilon, s}$ . If  $L \in \mathcal{D}^\prec \mathcal{B}^2$ , then  $\chi_{\epsilon, s}(L) \in \mathcal{D}^\prec G_{\epsilon, s}$ .*
- (b) *If  $L \in \mathcal{M}^\preceq \mathcal{B}^2$  and  $j > a + \nu + \rho$  then  $\chi_{\epsilon, s}^j(L) \in \mathcal{M}^\prec G_{\epsilon, s}$ .*

When  $\epsilon = 1, s = 0$  this is just [L12, 1.9]; the proof in the general case is entirely similar (it uses 1.4(a),(e)).



**1.8.** Let  $s \in \mathbf{Z}$ . For  $L \in \mathcal{C}_0^c \mathcal{B}^2$  we set

$$\underline{\chi}_{\epsilon,s}(L) = \underline{(\chi_{\epsilon,s}^{a+\nu+\rho}(L)((a+\nu+\rho)/2))} = \underline{(\chi_{\epsilon,s}(L))^{\{a+\nu+\rho\}}} \in \mathcal{C}_0^c G_{\epsilon,s}.$$

The functor  $\underline{\chi}_{\epsilon,s} : \mathcal{C}_0^c \mathcal{B}^2 \rightarrow \mathcal{C}_0^c G_{\epsilon,s}$  is called *truncated induction*. For  $z \in \mathbf{c}$  we have

$$(a) \quad \underline{\chi}_{\epsilon,s}(\mathbf{L}_z) = \underline{R_{\epsilon,s,z}^{n_z}(n_z/2)}.$$

When  $\epsilon = 1, s = 0$  this is just [L12, 1.10(a)]; the proof in the general case is entirely similar.

We shall denote by  $\tau : \mathbf{J}^c \rightarrow \mathbf{Z}$  the group homomorphism such that  $\tau(t_z) = 1$  if  $z \in \mathbf{D}_c$  and  $\tau(t_z) = 0$ , otherwise. For  $z, u \in \mathbf{c}$  we have:

$$(b) \quad \dim \operatorname{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_z), \underline{\chi}_{\epsilon,s}(\mathbf{L}_u)) = \sum_{y \in \mathbf{c}} \tau(t_{y^{-1}} t_z t_{\epsilon(y)} t_{u^{-1}}).$$

When  $\epsilon = 1, s = 0$  this is just [L12, 1.10(b)]. We now consider the general case.

Using (a) and the definitions we see that the left hand side of (b) equals

$$\sum_{A \in CS_{\epsilon,s,\mathbf{c}}} (A : R_{\epsilon,s,z}^{n_z})(A : R_{\epsilon,s,u}^{n_u}),$$

hence, using 1.5(a) it equals

$$\sum_{E, E' \in \operatorname{Irr}_{\epsilon,\mathbf{c}} W} (-1)^{|z|+|u|} \sum_{A \in CS_{\epsilon,s,\mathbf{c}}} c_{A,E,\tilde{e}} c_{A,E',\tilde{e}} \operatorname{tr}(\tilde{e} t_z, E_\infty) \operatorname{tr}(\tilde{e} t_u, E'_\infty).$$

Replacing in the last sum  $\sum_{A \in CS_{\epsilon,s,\mathbf{c}}} c_{A,E,\tilde{e}} c_{A,E',\tilde{e}}$  by 1 if  $E = E'$  and by 0 if  $E \neq E'$  (see [L1, 3.9] in the case  $s \neq 0$  and [L3, 13.12], [L6, 35.18(g)] in the case  $s = 0$ ) we obtain

$$\sum_{E \in \operatorname{Irr}_{\epsilon,\mathbf{c}} W} (-1)^{|z|+|u|} \operatorname{tr}(\tilde{e} t_z, E_\infty) \operatorname{tr}(\tilde{e} t_u, E_\infty).$$

This is equal to  $(-1)^{|z|+|u|}$  times the trace of the operator  $\xi \mapsto t_z \epsilon(\xi) t_{u^{-1}}$  on  $\mathbf{Q} \otimes \mathbf{J}^c$  (see [L6, 34.14(a), 34.17]). The last trace is equal to the sum over  $y \in \mathbf{c}$  of the coefficient of  $t_y$  in  $t_z t_{\epsilon(y)} t_{u^{-1}}$ ; this coefficient is equal to  $\tau(t_{y^{-1}} t_z t_{\epsilon(y)} t_{u^{-1}})$  since for  $y, y' \in \mathbf{c}$ ,  $\tau(t_{y'} t_y)$  is 1 if  $y' = y^{-1}$  and is 0 if  $y' \neq y^{-1}$  (see [L10, 20.1(b)]). Thus we have

$$\dim \operatorname{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_z), \underline{\chi}_{\epsilon,s}(\mathbf{L}_u)) = (-1)^{|u|+|z|} \sum_{y \in \mathbf{c}} \tau(t_{y^{-1}} t_z t_{\epsilon(y)} t_{u^{-1}}).$$

Since  $\dim \operatorname{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_z), \underline{\chi}_{\epsilon,s}(\mathbf{L}_u)) \in \mathbf{N}$  and  $\sum_{y \in \mathbf{c}} \tau(t_{y^{-1}} t_z t_{\epsilon(y)} t_{u^{-1}}) \in \mathbf{N}$ , it follows that (b) holds.

**Lemma 1.9.** *Let  $s \in \mathbf{N}$ . Let  $Y_1, Y_2$  be among  $G_{\epsilon, s}, \mathcal{B}^2$  and let  $\mathbf{X} \in \mathcal{D}_m^{\leq} Y_1$ . Let  $c, c'$  be integers and let  $\Phi : \mathcal{D}_m^{\leq} Y_1 \rightarrow \mathcal{D}_m^{\leq} Y_2$  be a functor which takes distinguished triangles to distinguished triangles, commutes with shifts, maps  $\mathcal{D}_m^{\leq} Y_1$  into  $\mathcal{D}_m^{\leq} Y_2$  and maps complexes of weight  $\leq i$  to complexes of weight  $\leq i$  (for any  $i$ ). Assume that (a), (b) below hold:*

$$(a) \quad (\Phi(\mathbf{X}_0))^h \in \mathcal{M}_m^{\leq} Y_2 \text{ for any } \mathbf{X}_0 \in \mathcal{M}_m^{\leq} Y_1 \text{ and any } h > c;$$

$$(b) \quad \mathbf{X} \text{ has weight } \leq 0 \text{ and } \mathbf{X}^i \in \mathcal{M}^{\leq} Y_1 \text{ for any } i > c'.$$

Then

$$(c) \quad (\Phi(\mathbf{X}))^j \in \mathcal{M}^{\leq} Y_2 \text{ for any } j > c + c',$$

and we have canonically

$$(d) \quad \underline{(\Phi(\underline{\mathbf{X}}^{\{c'\}}))}^{\{c\}} = \underline{(\Phi(\mathbf{X}))}^{\{c+c'\}}.$$

When  $\epsilon = 1, s = 0$  this is just [L12, 1.12]; the proof in the general case is entirely similar.

**1.10.** Let  $s \in \mathbf{Z}$ . Let  $L \in \mathcal{C}_0^c \mathcal{B}^2$ . We have  $\mathfrak{D}(L) \in \mathcal{C}_0^c \mathcal{B}^2$ . Moreover we have canonically:

$$(a) \quad \underline{\chi}_{\epsilon, s}(\mathfrak{D}(L)) = \mathfrak{D}(\underline{\chi}_{\epsilon, s}(L)).$$

When  $\epsilon = 1, s = 0$  this is just [L12, 1.13]; the proof in the general case is entirely similar.

## 2. TRUNCATED RESTRICTION

**2.1.** Recall that  $\epsilon \in \mathfrak{A}$  is fixed. In this section we fix  $s \in \mathbf{Z}$ . Let  $\pi, f$  be as in 1.3. Now  $K \mapsto \zeta_{\epsilon, s}(K) = f_! \pi^* K$  defines a functor  $\mathcal{D}_m(G_{\epsilon, s}) \rightarrow \mathcal{D}_m(\mathcal{B}^2)$ . (When  $\epsilon = 1, s = 0$ ,  $\zeta_{\epsilon, s}$  is the same as  $\zeta$  of [L12, 2.5].) For  $i \in \mathbf{Z}, K \in \mathcal{D}_m(G_{\epsilon, s})$  we write  $\zeta_{\epsilon, s}^i(K)$  instead of  $(\zeta_{\epsilon, s}(K))^i$ .

Let  $b : G_{\epsilon^{-1}, -s} \xrightarrow{\sim} G_{\epsilon, s}, b' : \mathcal{B}^2 \xrightarrow{\sim} \mathcal{B}^2$  be as in 1.3. From the definitions we see that for  $K \in \mathcal{D}_m(G_{\epsilon^{-1}, -s})$  we have

$$(a) \quad \zeta_{\epsilon, s}(b'_! K) = b'_! \zeta_{\epsilon^{-1}, -s}(K).$$

**Proposition 2.2.** *For any  $L \in \mathcal{D}_m(\mathcal{B}^2)$  we have*

$$(a) \quad \zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \simeq \{\oplus_{y \in W; |y|=k} L_y \bullet L \bullet L_{\epsilon(y)^{-1}} \otimes \mathfrak{L}[[2k - 2\nu]]; k \in \mathbf{N}\},$$

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \simeq$$

(b)

$$\{\oplus_{y \in W; |y|=k} L_y \bullet L \bullet L_{\epsilon(y)^{-1}} \otimes \mathfrak{L}[[2k - 2\nu - 2\rho]] \otimes \Lambda^d \mathcal{X}[[d]](d/2); k \in \mathbf{N}, d \in [0, \rho]\},$$

where  $\mathfrak{L}, \mathcal{X}$  are as in 0.3.

When  $\epsilon = 1, s = 0$  this is proved in [L12, 2.6]. The proof in the general case will be quite similar to that in the case  $\epsilon = 1, s = 0$ . Let

$$Y = \{(B_1, B_2, B_3, B_4, F) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times G_s; F(B_1) = B_4, F(B_2) = B_3\}.$$

For  $ij = 14$  or  $23$  we define  $h'_{ij} : Y \rightarrow X_{\epsilon,s}$  by  $(B_1, B_2, B_3, B_4, F) \mapsto (B_i, B_j, F)$  and  $h_{ij} : Y \rightarrow \mathcal{B}^2$  by  $(B_1, B_2, B_3, B_4, F) \mapsto (B_i, B_j)$ . We have  $\pi^* \pi! = h'_{14!} h'_{23}{}^*$  hence

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) = f_! \pi^* \pi! f^*(L) = f_! h'_{14!} h'_{23}{}^* f^*(L) = h_{14!} h_{23}^* L.$$

For  $k \in \mathbf{N}$  let  $Y^k = \cup_{y \in W; |y|=k} Y_y$  where

$$Y_y = \{(B_1, B_2, B_3, B_4, F) \in Y; (B_1, B_2) \in \mathcal{O}_y, (B_3, B_4) \in \mathcal{O}_{\epsilon(y)^{-1}}\}$$

and let  $Y_s^{\geq k} := \cup_{k' \geq k} Y_s^{k'}$ , an open subset of  $Y_s$ ; let  $h_{ij}^k : Y_s^k \rightarrow \mathcal{B}^2$ ,  $h_{ij}^{\geq k} : Y_s^{\leq k} \rightarrow \mathcal{B}^2$  be the restrictions of  $h_{ij}$ . For any  $k \in \mathbf{N}$  we have a distinguished triangle

$$(h_{14!}^{\geq k+1} h_{23}^{\geq k+1*} L), h_{14!}^{\geq k} h_{23}^{\geq k*} L, h_{14!}^k h_{23}^{k*} L).$$

It follows that we have

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \simeq \{h_{14!}^k h_{23}^{k*} L; k \in \mathbf{N}\}.$$

For  $k \in \mathbf{N}$  let  $Z^k = \cup_{y \in W; |y|=k} Z_y$  where

$$Z_y = \{(B_1, B_2, B_3, B_4) \in \mathcal{B}^4; (B_1, B_2) \in \mathcal{O}_y, (B_3, B_4) \in \mathcal{O}_{\epsilon(y)^{-1}}\};$$

for  $i, j \in [1, 4]$  we define  $\tilde{h}_{ij}^k : Z^k \rightarrow \mathcal{B}^2$  and  $\tilde{h}_{ij}^y : Z_y \rightarrow \mathcal{B}^2$  by  $(B_1, B_2, B_3, B_4) \mapsto (B_i, B_j)$ . We have an obvious morphism  $u : Y^k \rightarrow Z^k$ . The fibre of  $u$  at  $(B_1, B_2, B_3, B_4) \in Z^k$  can be identified with the set of all  $F \in G_{\epsilon,s}$  such that  $F(B_1) = B_4, F(B_2) = B_3$ . Since  $(B_1, B_2) \in \mathcal{O}_y, (B_3, B_4) \in \mathcal{O}_{\epsilon(y)^{-1}}$  for some

$y \in W$ , we can find  $\tilde{F} \in G_{\epsilon,s}$  such that  $\tilde{F}(B_1) = B_4, \tilde{F}(B_2) = B_3$ ; hence the fibre above can be identified with

$$\begin{aligned} & \{g \in G; \text{Ad}(g)\tilde{F}(B_1) = B_4, \text{Ad}(g)\tilde{F}(B_2) = B_3\} \\ & = \{g \in G; \text{Ad}(g)(B_4) = B_4, \text{Ad}(g)(B_3) = B_3\} = B_3 \cap B_4 \end{aligned}$$

which is quasi-isomorphic to  $\mathbf{k}^{\nu-k}$  times the  $\rho$ -dimensional torus  $T$ . We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{B}^2 & \xleftarrow{h_{23}^k} & Y_s^k & \xrightarrow{h_{14}^k} & \mathcal{B}^2 \\ 1 \downarrow & & u \downarrow & & 1 \downarrow \\ \mathcal{B}^2 & \xleftarrow{\tilde{h}_{23}^k} & Z^k & \xrightarrow{\tilde{h}_{14}^k} & \mathcal{B}^2 \end{array}$$

We have

$$h_{14!}^k h_{23}^{k*} L = \tilde{h}_{14!}^k u! u^* \tilde{h}_{23}^{k*} L = \tilde{h}_{14!}^k (\tilde{h}_{23}^{k*} L \otimes u! \bar{\mathbf{Q}}_l) = (\tilde{h}_{14!}^k \tilde{h}_{23}^{k*} L) \otimes \mathcal{L}[-2\nu + 2k].$$

We deduce that

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \simeq \{(\tilde{h}_{14!}^k \tilde{h}_{23}^{k*} L) \otimes \mathcal{L}[-2\nu + 2k]; k \in \mathbf{N}\}.$$

Since  $Z^k$  is the union of open and closed subvarieties  $Z_y, |y| = k$ , we have

$$\tilde{h}_{14!}^k \tilde{h}_{23}^{k*} L = \oplus_{y \in W; |y|=k} \tilde{h}_{14!}^y \tilde{h}_{23}^{y*} L.$$

From the definitions we have

$$\tilde{h}_{14!}^y \tilde{h}_{23}^{y*} L = L_y \bullet L \bullet L_{\epsilon(y)^{-1}}.$$

This completes the proof of (a). Now (b) follows from (a) just as in the case where  $\epsilon = 1, s = 0$ .

*In the remainder of this section we fix a two-sided cell  $\mathbf{c}$  of  $W$  such that  $\epsilon(\mathbf{c}) = \mathbf{c}$ ; we set  $a = \mathbf{a}(\mathbf{c})$ .*

**Proposition 2.3.** *Let  $w \in W$  and let  $j \in \mathbf{Z}$ . We set  $S = \zeta_{\epsilon,s}(R_{\epsilon,s,w})[[2\rho + 2\nu + |w|]] \in \mathcal{D}_m(\mathcal{B}^2)$ .*

- (a) *If  $w \preceq \mathbf{c}$  then  $S^j \in \mathcal{M}^{\preceq} \mathcal{B}^2$ .*
- (b) *If  $w \in \mathbf{c}$  and  $j > \nu + 2a$  then  $S^j \in \mathcal{M}^{\prec} \mathcal{B}^2$ .*
- (c) *If  $w \prec \mathbf{c}$  then  $S^j \in \mathcal{M}^{\prec} \mathcal{B}^2$ .*
- (d)  *$S^j$  is mixed of weight  $\leq j$ .*
- (e) *If  $j \neq \nu + 2a$  and  $w \in \mathbf{c}$  then  $gr_{\nu+2a} S^j \in \mathcal{M}^{\prec} \mathcal{B}^2$ .*
- (f) *If  $k > \nu + 2a$  and  $w \in \mathbf{c}$  then  $gr_k S^j \in \mathcal{M}^{\prec} \mathcal{B}^2$ .*

When  $\epsilon = 1, s = 0$  this is just [L12, 2.7]. The proof in the general case is entirely similar; it uses 2.2.

**Proposition 2.4.** (a) If  $K \in \mathcal{D}^{\preceq} G_{\epsilon,s}$  then  $\zeta_{\epsilon,s}(K) \in \mathcal{D}^{\preceq} \mathcal{B}^2$ . If  $K \in \mathcal{D}^{\prec} G_{\epsilon,s}$ , then  $\zeta_{\epsilon,s}(K) \in \mathcal{D}^{\prec} \mathcal{B}^2$ .

(b) If  $K \in \mathcal{M}^{\preceq} G_{\epsilon,s}$  and  $j > \rho + \nu + a$  then  $\zeta_{\epsilon,s}^j(K) \in \mathcal{M}^{\prec} \mathcal{B}^2$ .

When  $\epsilon = 1, s = 0$  this is just [L12, 2.8]. The proof in the general case is entirely similar; it uses 1.5(b) and 2.3.

**2.5.** For  $K \in \mathcal{C}_0^c G_{\epsilon,s}$  we set

$$\underline{\zeta}_{\epsilon,s}(K) = \underline{(\zeta_{\epsilon,s}(K))^{\{\rho+\nu+a\}}} \in \mathcal{C}_0^c \mathcal{B}^2.$$

We say that  $\underline{\zeta}_{\epsilon,s}(K)$  is the *truncated restriction* of  $K$ .

**Proposition 2.6.** Let  $K \in \mathcal{D}_m(G_{\epsilon,s})$  and let  $L \in \mathcal{C}_0^\spadesuit \mathcal{B}^2$ . Then there is a canonical isomorphism  ${}^\epsilon L \bullet \zeta_{\epsilon,s}(K) \xrightarrow{\sim} \zeta_{\epsilon,s}(K) \bullet L$ .

When  $\epsilon = 1, s = 0$  this follows from [L12, 2.10(a)]. We now consider the general case. Let  $u : G_{\epsilon,s} \times \mathcal{B}^2 \rightarrow G_{\epsilon,s} \times \mathcal{B}^2$  be as in 1.6(a). We have  $\zeta_{\epsilon,s}(K) \bullet L = c_! d^*(K \boxtimes L)$  where  $Z = \{(F, (B, B'', B')) \in G_{\epsilon,s} \times \mathcal{B}^3; F(B) = B''\}$ ,  $d : Z \rightarrow G_{\epsilon,s} \times \mathcal{B}^2$  is  $(F, (B, B'', B')) \mapsto (F, (B'', B'))$ ,  $c : Z \rightarrow \mathcal{B}^2$  is  $(F, (B, B'', B')) \mapsto (B, B')$ . We have  ${}^\epsilon L \bullet \zeta_{\epsilon,s}(K) = c'_! d'^*(K \boxtimes {}^\epsilon L)$  where  $Z' = \{(F, (B, B'', B')) \in G_{\epsilon,s} \times \mathcal{B}^3; F(B'') = B'\}$ ,  $d' : Z' \rightarrow G_{\epsilon,s} \times \mathcal{B}^2$  is  $(F, (B, B'', B')) \mapsto (F, (B, B''))$ ,  $c' : Z' \rightarrow \mathcal{B}^2$  is  $(F, (B, B'', B')) \mapsto (B, B')$ . Using 1.6(a) we have  $K \boxtimes {}^\epsilon L = u^*(K \boxtimes L)$  hence it is enough to show that  $c_! d^*(K \boxtimes L) = c'_! d'^* u^*(K \boxtimes L)$ . We have  $c_! d^*(K \boxtimes L) = c_{1!} d_1^*(K \boxtimes L) = c'_! d'^* u^*(K \boxtimes L)$  where  $d_1 : G_{\epsilon,s} \times \mathcal{B}^2 \rightarrow G_{\epsilon,s} \times \mathcal{B}^2$  is  $(F, (B, B')) \mapsto (F, (F(B), B'))$ ,  $c_1 : G_{\epsilon,s} \times \mathcal{B}^2 \rightarrow \mathcal{B}^2$  is  $(F, (B, B')) \mapsto (B, B')$ . The proposition follows.

**Proposition 2.7.** (a) If  $L \in \mathcal{M}^{\preceq} \mathcal{B}^2$  and  $j > 2a + 2\nu + 2\rho$  then  $(\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))^j \in \mathcal{M}^{\prec} \mathcal{B}^2$ .

(b) If  $L \in \mathcal{C}_0^c \mathcal{B}^2$ , we have canonically

$$\underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L)) = \underline{(\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))^{\{2a+2\nu+2\rho\}}} \in \mathcal{C}_0^c \mathcal{B}^2.$$

We apply 1.9 with  $\Phi = \zeta_{\epsilon,s} : \mathcal{D}_m(G_{\epsilon,s}) \rightarrow \mathcal{D}_m(\mathcal{B}^2)$  and with  $\mathbf{X} = \chi_{\epsilon,s}(L)$ ,  $(c, c') = (a + \nu + \rho, a + \nu + \rho)$ , see 2.4, 1.7. The result follows.

**2.8.** For  $L, L' \in \mathcal{C}_0^c \mathcal{B}^2$ , we set (as in [L12, 3.2])

$$(a) \quad L \bullet L' = \underline{(L \bullet L')^{\{a-\nu\}}} \in \mathcal{C}_0^c \mathcal{B}^2.$$

This defines an associative tensor product structure on  $\mathcal{C}_0^c \mathcal{B}^2$ .

**Proposition 2.9.** *Let  $K \in \mathcal{C}_0^c G_{\epsilon,s}$ ,  $L \in \mathcal{C}_0^c \mathcal{B}^2$ . There is a canonical isomorphism*

$$(a) \quad {}^\epsilon L \bullet \zeta_{\epsilon,s}(K) \xrightarrow{\sim} \zeta_{\epsilon,s}(K) \bullet L.$$

Applying 1.9 with  $\Phi : \mathcal{D}_m^{\leq} \mathcal{B}^2 \rightarrow \mathcal{D}_m^{\leq} \mathcal{B}^2$ ,  $L' \mapsto L' \bullet L$ ,  $\mathbf{X} = \zeta_{\epsilon,s}(K)$ ,  $(c, c') = (a - \nu, a + \rho + \nu)$  (see [L12, 3.1] and 2.4), we deduce that we have canonically

$$(b) \quad \underline{((\zeta_{\epsilon,s}(K))^{\{a+\rho+\nu\}} \bullet L)^{\{a-\nu\}}} = \underline{(\zeta_{\epsilon,s}(K) \bullet L)^{\{2a+\rho\}}}.$$

Using 1.9 with  $\Phi : \mathcal{D}_m^{\leq} \mathcal{B}^2 \rightarrow \mathcal{D}_m^{\leq} \mathcal{B}^2$ ,  $L' \mapsto {}^\epsilon L \bullet L'$ ,  $\mathbf{X} = \zeta_{\epsilon,s}(K)$ ,  $(c, c') = (a - \nu, a + \rho + \nu)$  (see [L12, 3.1] and 2.8), we deduce that we have canonically

$$(c) \quad \underline{({}^\epsilon L \bullet (\zeta_{\epsilon,s}(K))^{\{a+\rho+\nu\}})^{\{a-\nu\}}} = \underline{({}^\epsilon L \bullet \zeta_{\epsilon,s}(K))^{\{2a+\rho\}}}.$$

We now combine (b),(c) with 2.6; we obtain the isomorphism (a).

**2.10.** Define  $c : G_{\epsilon,s} \times \mathcal{B}^2 \rightarrow \mathcal{B}^2$  by  $(F, B, B') \mapsto (F(B), F(B'))$ . We show that for  $K \in \mathcal{C}^\spadesuit G_{\epsilon,s}$  we have canonically

$$(a) \quad c^* \zeta_{\epsilon,s} K = \bar{\mathbf{Q}}_l \boxtimes \zeta_{\epsilon,s} K.$$

We have a commutative diagram with cartesian left squares

$$\begin{array}{ccccccc} G_{\epsilon,s} \times \mathcal{B}^2 & \xleftarrow{f''} & X''_{\epsilon,s} & \xrightarrow{\pi''} & G_{\epsilon,s} \times G_{\epsilon,s} & \xrightarrow{e} & G_{\epsilon,s} \\ d \downarrow & & d' \downarrow & & d'' \downarrow & & \\ & & G_{\epsilon,s} \times \mathcal{B}^2 & \xleftarrow{f'} & X'_{\epsilon,s} & \xrightarrow{\pi'} & G_{\epsilon,s} \downarrow c' \downarrow \\ \mathcal{B}^2 & \xleftarrow{f} & X_{\epsilon,s} & \xrightarrow{\pi} & G_{\epsilon,s} & & \end{array}$$

where  $f, g$  are as in 1.3,

$$X'_{\epsilon,s} = \{(\tilde{F}, B, B', F) \in G_{\epsilon,s} \times \mathcal{B} \times \mathcal{B} \times G_{\epsilon,s}; F\tilde{F}(B) = \tilde{F}(B')\},$$

$$X''_{\epsilon,s} = \{(\tilde{F}, B, B', F) \in G_{\epsilon,s} \times \mathcal{B} \times \mathcal{B} \times G_{\epsilon,s}; F(B) = B'\},$$

$$f'(\tilde{F}, B, B', F) = (\tilde{F}, B, B'), f''(\tilde{F}, B, B', F) = (\tilde{F}, B, B'),$$

$$\pi'(\tilde{F}, B, B', F) = F, \pi''(\tilde{F}, B, B', F) = (\tilde{F}, F),$$

$$c'(\tilde{F}, B, B', F) = (F, \tilde{F}(B), \tilde{F}(B')), c''(F) = F, d(\tilde{F}, B, B') = (\tilde{F}, B, B'),$$

$$d'(\tilde{F}, B, B', F) = (\tilde{F}, \tilde{F}^{-1} F \tilde{F}, B, B'), d''(\tilde{F}, F) = \tilde{F}^{-1} F \tilde{F}, e(\tilde{F}, F) = F.$$

It is enough to show that  $d^* f'_1 c'^* \pi^* K = f''_1 \pi''^* e^* K$ . or that  $f'_1 d'^* \pi'^* K = f''_1 \pi''^* e^* K$ . ■

It is enough to show that  $d'^* \pi'^* K = \pi''^* e^* K$ , or that  $\pi''^* d''^* K = \pi''^* e^* K$ . Hence it is enough to show that  $d''^* K = e^* K$ . We identify  $G \times G_{\epsilon,s} \leftrightarrow G_{\epsilon,s} \times G_{\epsilon,s}$  by  $(g, F) \leftrightarrow (F \text{Ad}(g), F)$ . Then  $d'', e : G_{\epsilon,s} \times G_{\epsilon,s} \rightarrow G_{\epsilon,s}$  become the maps

$d_1, e_1 : G \times G_{\epsilon, s} \rightarrow G_{\epsilon, s}$  given by  $(g, F) \mapsto \text{Ad}(g)(^{-1}F\text{Ad}(g))$ ,  $(g, F) = F$  respectively and we have  $d_1^*K = e_1^*K$  by the  $G$ -equivariance of  $K$ . Hence  $d''^*K = e^*K$  as required.

Using (a) and the definitions we see that for any  $K \in \mathcal{C}_0^{\mathfrak{c}}G_{\epsilon, s}$  we have canonically

$$(b) \quad c^* \zeta_{\epsilon, s} K = \bar{\mathbf{Q}}_l \boxtimes \zeta_{\epsilon, s} K.$$

From the definitions (see 1.6) for any  $L \in \mathcal{C}_0^{\spadesuit} \mathcal{B}^2$  we have  $c^*L = \bar{\mathbf{Q}}_l \boxtimes^{\epsilon} L$ . Comparing with (b) we deduce that we have canonically

$$(c) \quad \epsilon(\zeta_{\epsilon, s} K) = \zeta_{\epsilon, s} K$$

for any  $K \in \mathcal{C}_0^{\mathfrak{c}}G_{\epsilon, s}$ .

### 3. TRUNCATED CONVOLUTION FROM $G_{\epsilon, s} \times G_{\epsilon', s'}$ TO $G_{\epsilon\epsilon', s+s'}$

**3.1.** Let  $\epsilon, \epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . We define  $\mu : G_{\epsilon, s} \times G_{\epsilon', s'} \rightarrow G_{\epsilon\epsilon', s+s'}$  by  $(F, F') = FF'$  (composition of maps  $G \rightarrow G$ ); this is a quasi-morphism, see 1.3. For  $K \in \mathcal{D}_m(G_{\epsilon, s})$ ,  $K' \in \mathcal{D}_m(G_{\epsilon', s'})$  we define the convolution  $K * K' \in \mathcal{D}_m(G_{\epsilon\epsilon', s+s'})$  by  $K * K' = \mu_!(K \boxtimes K')$ . If  $\epsilon'' \in \mathfrak{A}$ ,  $s'' \in \mathbf{Z}$  then for  $K, K'$  as above and  $K'' \in \mathcal{D}_m(G_{\epsilon'', s''})$ , we have canonically  $(K * K') * K'' = K * (K' * K'') \in \mathcal{D}_m(G_{\epsilon\epsilon'\epsilon'', s+s'+s''})$  (and we denote this by  $K * K' * K''$ ).

**Lemma 3.2.** *Let  $\epsilon, \epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . Let  $K \in \mathcal{D}_m(G_{\epsilon, s})$ ,  $L \in \mathcal{D}_m(\mathcal{B}^2)$ . We have canonically  $K * \chi_{\epsilon', s'}(L) = \chi_{\epsilon\epsilon', s+s'}(L \bullet \zeta_{\epsilon, s}(K))$ .*

Let

$$Z = \{(F_1, F_2, B, B') \in G_{\epsilon, s} \times G_{\epsilon', s'} \times \mathcal{B} \times \mathcal{B}; F_2(B) = B'\}.$$

Define  $c : Z \rightarrow G_{\epsilon, s} \times \mathcal{B}^2$  by  $(F_1, F_2, B, B') \mapsto (F_1, (B, B'))$  and  $d : Z \rightarrow G_{\epsilon\epsilon', s+s'}$  by  $(F_1, F_2, B, B') \mapsto F_1 F_2$ . From the definitions we see that both

$$K * \chi_{\epsilon', s'}(L), \chi_{\epsilon\epsilon', s+s'}(L \bullet \zeta_{\epsilon, s}(K))$$

can be identified with  $d_! c^*(K \boxtimes L)$ . The lemma follows. (In the case where  $\epsilon = \epsilon' = 1$  and  $s = s' = 0$  this reduces to [L12, 4.2].)

**Proposition 3.3.** *Let  $\epsilon, \epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . For any  $L, L' \in \mathcal{D}_m(\mathcal{B}^2)$  we have*

$$\begin{aligned} & \chi_{\epsilon, s}(L) * \chi_{\epsilon', s'}(L') [[2\rho + 2\nu]] \\ & \simeq \{\chi_{\epsilon\epsilon', s+s'}(L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}}) [[2|y|]] \otimes \Lambda^d \mathcal{X}[[d]](d/2); d \in [0, \rho], y \in W\}. \end{aligned}$$

From 2.2(b) we deduce

$$\begin{aligned} & L' \bullet \zeta_{\epsilon, s}(\chi_{\epsilon, s}(L)) [[2\nu + 2\rho]] \\ & \simeq \{L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}} [[2|y|]] \otimes \Lambda^d \mathcal{X}[[d]](d/2); y \in W, d \in [0, \rho]\} \end{aligned}$$

and

$$\begin{aligned} & \chi_{\epsilon\epsilon',s+s'}(L' \bullet \zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))[[2\nu + 2\rho]] \\ & \simeq \{\chi_{\epsilon\epsilon',s+s'}(L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}})[[2|y|]] \otimes \Lambda^d \mathcal{X}[[d]](d/2); y \in W, d \in [0, \rho]\}. \end{aligned}$$

It remains to show that  $\chi_{\epsilon\epsilon',s+s'}(L' \bullet \zeta_{\epsilon,s}(\chi_{\epsilon,s}(L))) = \chi_{\epsilon,s}(L) * \chi_{\epsilon',s'}(L')$ . This follows from 3.2 with  $K, L$  replaced by  $\chi_{\epsilon,s}(L), L'$ .

In the remainder of this section we fix a two-sided cell  $\mathbf{c}$  of  $W$ ; we set  $a = \mathbf{a}(\mathbf{c})$ .

**Proposition 3.4.** *Let  $\epsilon, \epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . Assume that  $\epsilon(\mathbf{c}) = \mathbf{c}$ ,  $\epsilon'(\mathbf{c}) = \mathbf{c}$ . Let  $w, w' \in W$  and let  $j \in \mathbf{Z}$ . We set  $C = R_{\epsilon,s,w} * R_{\epsilon',s',w'}[[2\rho + 2\nu + |w| + |w'|]] \in \mathcal{D}_m(G_{\epsilon\epsilon',s+s'})$ .*

- (a) *If  $w \preceq \mathbf{c}$  or  $w' \preceq \mathbf{c}$  then  $C^j \in \mathcal{M}^{\preceq} G_{\epsilon\epsilon',s+s'}$ .*
- (b) *If  $j > \Delta + 4a$  and either  $w \in \mathbf{c}$  or  $w' \in \mathbf{c}$  then  $C^j \in \mathcal{M}^{\prec} G_{\epsilon\epsilon',s+s'}$ .*
- (c) *If  $w \prec \mathbf{c}$  or  $w' \prec \mathbf{c}$  then  $C^j \in \mathcal{M}^{\prec} G_{\epsilon\epsilon',s+s'}$ .*
- (d)  *$C^j$  is mixed of weight  $\leq j$ .*
- (e) *If  $j \neq \Delta + 4a$  and either  $w \in \mathbf{c}$  or  $w' \in \mathbf{c}$  then  $\text{gr}_{\Delta+4a} C^j \in \mathcal{M}^{\prec} G_{\epsilon\epsilon',s+s'}$ .*
- (f) *If  $k > \Delta + 4a$  and  $w \in \mathbf{c}$  or  $w' \in \mathbf{c}$  then  $\text{gr}_k C^j \in \mathcal{M}^{\prec} G_{\epsilon\epsilon',s+s'}$ .*

When  $\epsilon = \epsilon' = 1$ ,  $s = s' = 0$ , this is just [L12, 4.4]. The proof in the general case is entirely similar; it uses 3.3 and 1.4(d),(e).

**Proposition 3.5.** *Let  $\epsilon, \epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . Assume that  $\epsilon(\mathbf{c}) = \mathbf{c}$ ,  $\epsilon'(\mathbf{c}) = \mathbf{c}$ . Let  $K \in \mathcal{D}_m^{\blacklozenge}(G_{\epsilon,s})$ ,  $K' \in \mathcal{D}_m^{\blacklozenge}(G_{\epsilon',s'})$ .*

- (a) *If  $K \in \mathcal{D}^{\preceq} G_{\epsilon,s}$  or  $K' \in \mathcal{D}^{\preceq} G_{\epsilon',s'}$  then  $K * K' \in \mathcal{D}^{\preceq} G_{\epsilon\epsilon',s+s'}$ ; if  $K \in \mathcal{D}^{\prec} G_{\epsilon,s}$  or  $K' \in \mathcal{D}^{\prec} G_{\epsilon',s'}$  then  $K * K' \in \mathcal{D}^{\prec} G_{\epsilon\epsilon',s+s'}$ .*
- (b) *If  $K \in \mathcal{M}^{\preceq} G_{\epsilon,s}$ ,  $K' \in \mathcal{M}^{\preceq} G_{\epsilon',s'}$  and  $j > \rho + 2a$  then  $(K * K')^j \in \mathcal{M}^{\prec} G_{\epsilon\epsilon',s+s'}$ .*

When  $\epsilon = \epsilon' = 1$ ,  $s = s' = 0$ , this is just [L12, 4.5]. The proof in the general case is entirely similar; it uses 3.4.

**3.6.** Let  $\epsilon, \epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . Assume that  $\epsilon(\mathbf{c}) = \mathbf{c}$ ,  $\epsilon'(\mathbf{c}) = \mathbf{c}$ . For  $K \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon,s}$ ,  $K' \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon',s'}$  we set

$$K \underline{*} K' = \underline{(K * K')}^{\{2a+\rho\}} \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon\epsilon',s+s'}.$$

We say that  $K \underline{*} K'$  is the *truncated convolution* of  $K, K'$ .

**Proposition 3.7.** *Let  $\epsilon, \epsilon', \epsilon'' \in \mathfrak{A}$ ,  $s, s', s'' \in \mathbf{Z}$ . Assume that  $\epsilon(\mathbf{c}) = \mathbf{c}$ ,  $\epsilon'(\mathbf{c}) = \mathbf{c}$ ,  $\epsilon''(\mathbf{c}) = \mathbf{c}$ . Let  $K, K', K''$  be in  $\mathcal{C}_0^{\mathbf{c}} G_{\epsilon,s}, \mathcal{C}_0^{\mathbf{c}} G_{\epsilon',s'}, \mathcal{C}_0^{\mathbf{c}} G_{\epsilon'',s''}$  respectively. There is a canonical isomorphism*

$$(a) \quad (K \underline{*} K') \underline{*} K'' \xrightarrow{\sim} K \underline{*} (K' \underline{*} K'').$$

When  $\epsilon = \epsilon' = \epsilon'' = 1$ ,  $s = s' = s'' = 0$ , this is just [L12, 4.7]. The proof in the general case is entirely similar; it uses 1.9, 3.5.



**Proposition 3.8.** *Let  $\epsilon, \epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . Assume that  $\epsilon(\mathbf{c}) = \mathbf{c}$ ,  $\epsilon'(\mathbf{c}) = \mathbf{c}$ . Let  $K \in \mathcal{C}_0^\epsilon G_{\epsilon, s}$ ,  $K' \in \mathcal{C}_0^{\epsilon'} G_{\epsilon', s'}$ . There is a canonical isomorphism (in  $\mathcal{C}_0^\epsilon \mathcal{B}^2$ ):*

$$\zeta_{\epsilon', s'}(K') \bullet \zeta_{\epsilon, s}(K) \xrightarrow{\sim} \zeta_{\epsilon\epsilon', s+s'}(K * K').$$

When  $\epsilon = \epsilon' = 1$ ,  $s = s' = 0$  this is just [L12, 5.2]. The proof in the general case is entirely similar.

#### 4. ANALYSIS OF THE COMPOSITION $\zeta_{\epsilon, s} \chi_{\epsilon, s}$

**4.1.** In the remainder of this paper we fix a two-sided cell  $\mathbf{c}$  of  $W$ ; we set  $a = \mathbf{a}(\mathbf{c})$ . We also fix  $\epsilon \in \mathfrak{A}$  such that  $\epsilon(\mathbf{c}) = \mathbf{c}$ . In this section we fix  $s \in \mathbf{Z}$ . Let  $e, f, e'$  be integers such that  $e \leq f \leq e' - 3$  and let  $\mathbf{e} = e' - e + 1$ ; we have  $\mathbf{e} \geq 4$ . We set

$$\mathcal{Y} = \{((B_e, B_{e+1}, \dots, B_{e'}), F) \in \mathcal{B}^{\mathbf{e}} \times G_s; F(B_f) = B_{f+3}, F(B_{f+1}) = B_{f+2}\}.$$

Define  $\vartheta : \mathcal{Y} \rightarrow \mathcal{B}^{\mathbf{e}}$  by  $((B_e, B_{e+1}, \dots, B_{e'}), F) \mapsto (B_e, B_{e+1}, \dots, B_{e'})$ . For  $i, j$  in  $\{e, e+1, \dots, e'\}$  let  $p_{ij} : \mathcal{B}^{\mathbf{e}} \rightarrow \mathcal{B}^2$  be the projection to the  $i, j$  coordinate; define  $h_{ij} : \mathcal{Y} \rightarrow \mathcal{B}^2$  by  $h_{ij} = p_{ij}\vartheta$ . Now  $G^{\mathbf{e}-2}$  acts on  $\mathcal{Y}$  by

$$\begin{aligned} (g_e, \dots, g_f, g_{f+3}, \dots, g_{e'}) : ((B_e, B_{e+1}, \dots, B_{e'}), F) \mapsto \\ (\text{Ad}(g_e)(B_e), \text{Ad}(g_{e+1})(B_{e+1}), \dots, \text{Ad}(g_{f-1})(B_{f-1}), \text{Ad}(g_f)(B_f), \text{Ad}(g_f)(B_{f+1}), \\ \text{Ad}(g_{f+3})(B_{f+2}), \text{Ad}(g_{f+3})(B_{f+3}), \text{Ad}(g_{f+4})(B_{f+4}), \dots, \text{Ad}(g_{e'})(B_{e'})), \\ \text{Ad}(g_{f+3})F\text{Ad}(g_f^{-1})); \end{aligned}$$

this induces a  $G^{\mathbf{e}-2}$ -action on  $\mathcal{B}^{\mathbf{e}}$  so that  $\vartheta$  is  $G^{\mathbf{e}-2}$ -equivariant.

Let  $E = \{e, e+1, \dots, e'-1\} - \{f, f+2\}$ . Assume that  $x_n \in \mathbf{c}$  are given for  $n \in E$ . Let  $P = \otimes_{n \in E} p_{n, n+1}^* \mathbf{L}_{x_n} \in \mathcal{D}_m \mathcal{B}^{\mathbf{e}}$ ,  $\tilde{P} = \otimes_{n \in E} h_{n, n+1}^* \mathbf{L}_{x_n} = \vartheta^* P \in \mathcal{D}_m \mathcal{Y}$ . In 4.1-4.7 we will study

$$h_{ee'!} \tilde{P} \in \mathcal{D}_m \mathcal{B}^2.$$

Setting  $\Xi = \vartheta_! \bar{\mathbf{Q}}_l \in \mathcal{D}_m \mathcal{B}^{\mathbf{e}}$ , we have

$$h_{ee'!} \tilde{P} = p_{ee!}(\Xi \otimes P).$$

Clearly,  $\Xi^j$  is  $G^{\mathbf{e}-2}$ -equivariant for any  $j$ . For any  $y, y'$  in  $W$  we set

$$Z_{y, y'} := \{(B_e, B_{e+1}, \dots, B_{e'}) \in \mathcal{B}^{\mathbf{e}}; (B_f, B_{f+1}) \in \mathcal{O}_y, (B_{f+2}, B_{f+3}) \in \mathcal{O}_{y'}\}.$$

These are the orbits of the  $G^{\mathbf{e}-2}$ -action on  $\mathcal{B}^{\mathbf{e}}$ . Note that the fibre of  $\vartheta$  over a point of  $Z_{y, y'}$  is isomorphic to  $T \times \mathbf{k}^{\nu - |y|}$  if  $y' = \epsilon(y)^{-1}$  and is empty if  $y' \neq \epsilon(y)^{-1}$ . Thus

- (a)  $\Xi|_{Z_{y, y'}}$  is 0 if  $y' \neq \epsilon(y)^{-1}$
- and for any  $y \in W$  we have
- (b)  $\mathcal{H}^h \Xi|_{Z_{y, \epsilon(y)^{-1}}} = 0$  if  $h > 2\nu - 2|y| + 2\rho$ ,  $\mathcal{H}^{2\nu - 2|y| + 2\rho} \Xi|_{Z_{y, \epsilon(y)^{-1}}} = \bar{\mathbf{Q}}_l(-\nu + |y| - \rho)$ .

The closure of  $Z_{y, y'}$  in  $\mathcal{B}^{\mathbf{e}}$  is denoted by  $\bar{Z}_{y, y'}$ . We set  $k_{\mathbf{e}} = \mathbf{e}\nu + 2\rho$ . We have the following result.

**Lemma 4.2.** (a) We have  $\Xi^j = 0$  for any  $j > k_{\mathbf{e}}$ . Hence, setting  $\Xi' = \tau_{\leq k_{\mathbf{e}}-1} \Xi$ , we have a canonical distinguished triangle  $(\Xi', \Xi, \Xi^{k_{\mathbf{e}}}[-k_{\mathbf{e}}])$ .

(b) If  $\xi \in Z_{y,y'}$  and  $i = 2\nu - |y| - |y'| + 2\rho$ , the induced homomorphism  $\mathcal{H}_{\xi}^i \Xi \rightarrow \mathcal{H}_{\xi}^{i-k_{\mathbf{e}}}(\Xi^{k_{\mathbf{e}}})$  is an isomorphism.

When  $\epsilon = 1, s = 0$  this is just [L12, 6.2]. The proof in the general case is entirely similar; it uses 4.1(a),(b).

**4.3.** For any  $y, y'$  in  $W$  let  $\mathfrak{T}_{y,y'}$  be the intersection cohomology complex of  $\bar{Z}_{y,y'}$  extended by 0 on  $\mathcal{B}^{\mathbf{e}} - \bar{Z}_{y,y'}$ , to which  $[(\mathbf{e} - 2)\nu + |y| + |y'|]$  is applied. Note that

$$(a) \quad \mathfrak{T}_{y,y'} = p_{f,f+1}^* \mathbf{L}_y \otimes p_{f+2,f+3}^* \mathbf{L}_{y'} [(\mathbf{e} - 4)\nu].$$

We have the following result.

**Lemma 4.4.** We have canonically  $gr_0(\Xi^{k_{\mathbf{e}}}(k_{\mathbf{e}}/2)) = \oplus_{y \in W} \mathfrak{T}_{y, \epsilon(y)^{-1}}$ .

When  $\epsilon = 1, s = 0$  this is just [L12, 6.4]. The proof in the general case is entirely similar; it uses 4.2(b) and 4.1.

**4.5.** Let  $y, \tilde{y} \in W$ . Using the definitions and 1.2(a) we have

$$\begin{aligned} & p_{ee'}!(\mathfrak{T}_{y,\tilde{y}} \otimes P[(6 - 2\mathbf{e})\nu]) \\ (a) \quad & = L_{x_1}^{\#} \bullet \dots \bullet L_{x_{f-1}}^{\#} \bullet L_y^{\#} \bullet L_{x_{f+1}}^{\#} \bullet L_{\tilde{y}}^{\#} \bullet L_{x_{f+3}}^{\#} \bullet \dots \bullet L_{x_{e'}}^{\#} [[\nu + |y| + |\tilde{y}| + \sum_{n \in E} |x_n|]]. \end{aligned}$$

**Lemma 4.6.** The map  $\Xi \rightarrow \Xi^{k_{\mathbf{e}}}[-k_{\mathbf{e}}]$  (coming from  $(\Xi', \Xi, \Xi^{k_{\mathbf{e}}}[-k_{\mathbf{e}}])$  in 4.2(a)) induces a morphism

$$(p_{ee'}!(\Xi \otimes P))^{(\mathbf{e}-2)a+(6-\mathbf{e})\nu+2\rho} \rightarrow (p_{ee'}!(\Xi^{k_{\mathbf{e}}} \otimes P))^{(\mathbf{e}-2)a+(6-\mathbf{e})\nu+2\rho-k_{\mathbf{e}}}$$

whose kernel and cokernel are in  $\mathcal{M}_m^{\prec} \mathcal{B}^2$ .

When  $\epsilon = 1, s = 0$  this is just [L12, 6.6]. The proof in the general case is entirely similar; it uses 4.5(a),(b) and [L12, 2.2(a)].

**Lemma 4.7.** We have canonically

$$\underline{(h_{ee'}! \tilde{P})^{\{(\mathbf{e}-2)a+(6-\mathbf{e})\nu+2\rho\}}} = \oplus_{y \in \mathbf{c}} Q_y$$

where

$$\begin{aligned} Q_y &= \underline{(p_{ee'}!(\mathfrak{T}_{y, \epsilon(y)^{-1}} \otimes P))^{\{(\mathbf{e}-2)a+(6-2\mathbf{e})\nu\}}} \\ &= \mathbf{L}_{x_1} \bullet \dots \bullet \mathbf{L}_{x_{f-1}} \bullet \mathbf{L}_y \bullet \mathbf{L}_{x_{f+1}} \bullet \mathbf{L}_{\epsilon(y)^{-1}} \bullet \mathbf{L}_{x_{f+3}} \bullet \dots \bullet \mathbf{L}_{x_{e'}}. \end{aligned}$$

When  $\epsilon = 1, s = 0$  this is just [L12, 6.7]. The proof in the general case is entirely similar; it uses 4.6, 4.5(a),(b) and [L12, 2.2(a), 2.3, 3.2].

**Theorem 4.8.** *Let  $x \in \mathbf{c}$ . We have canonically*

$$(a) \quad \zeta_{\epsilon,s}(\chi_{\epsilon,s}(\mathbf{L}_x)) = \oplus_{y \in \mathbf{c}} \mathbf{L}_y \bullet \mathbf{L}_x \bullet \mathbf{L}_{\epsilon(y)^{-1}}.$$

When  $\epsilon = 1, s = 0$  this is just [L12, 6.8]. The proof in the general case is entirely similar; it uses 4.7, the proof of 2.2 and 2.7(b).

**4.9.** Using [L12, 2.4] we see that 4.8(a) implies

$$(a) \quad \zeta_{\epsilon,s} \chi_{\epsilon,s} \mathbf{L}_x \cong \oplus_{z \in \mathbf{c}} (\mathbf{L}_z)^{\oplus \psi_x(z)}$$

in  $\mathcal{C}^c \mathcal{B}^2$  where  $\psi_x(z) \in \mathbf{N}$  are given by the following equation in  $\mathbf{J}^c$ :

$$\sum_{y \in \mathbf{c}} t_y t_x t_{\epsilon(y)^{-1}} = \sum_{z \in \mathbf{c}} \psi_x(z) t_z.$$

## 5. ADJUNCTION FORMULA (WEAK FORM)

**Proposition 5.1.** *Let  $\epsilon' \in \mathfrak{A}$ ,  $s, s' \in \mathbf{Z}$ . We assume that  $\epsilon'(\mathbf{c}) = \mathbf{c}$ . Let  $K \in \mathcal{C}_0^c(G_{\epsilon,s})$ ,  $L \in \mathcal{C}_0^c(\mathcal{B}^2)$ . We have canonically*

$$(a) \quad K * \chi_{\epsilon',s'}(L) = \chi_{\epsilon\epsilon',s+s'}(L \bullet \zeta_{\epsilon,s}(K)).$$

When  $\epsilon = \epsilon' = 1, s = s' = 0$  this is just [L12, 8.1]. The proof in the general case is entirely similar; it uses 3.2.

**5.2.** Let  $s \in \mathbf{Z}$ . When  $\epsilon = 1, s = 0$ , the arguments in this subsection reduce to arguments in [L12, 8.8]. Let  $u' : G_{\epsilon^{-1},-s} \rightarrow \mathbf{p}$  be the obvious map. From [L2, 7.4] we see that for  $K, K' \in \mathcal{M}_m^{\preceq} G_{\epsilon^{-1},-s}$  we have canonically

$$(u'_!(K \otimes K'))^0 = \text{Hom}_{\mathcal{M}G_{\epsilon^{-1},-s}}(\mathfrak{D}(K), K'), \quad (u'_!(K \otimes K'))^j = 0 \text{ if } j > 0.$$

We deduce that if  $K, K'$  are also pure of weight 0 then  $(u'_!(K \otimes K'))^0$  is pure of weight zero that is,  $(u'_!(K \otimes K'))^0 = \text{gr}_0(u'_!(K \otimes K'))^0$ . Let  $\iota : \mathbf{p} \rightarrow G = G_0$  be the map with image 1. From the definitions we see that we have  $u'_!(K \otimes K') = \iota^*(b_!(K) * K')$  where  $b : G_{\epsilon^{-1},-s} \rightarrow G_{\epsilon,s}$  is given by  $F \mapsto F^{-1}$ . Hence for  $K, K' \in \mathcal{C}_0^c G_{\epsilon^{-1},-s}$  we have

$$(a) \quad \text{Hom}_{\mathcal{C}^c G_{\epsilon^{-1},-s}}(\mathfrak{D}(K), K') = (\iota^*(b_!(K) * K'))^0 = (\iota^*(b_!(K) * K'))^{\{0\}}.$$

Applying [L12, 8.2] with  $\Phi : \mathcal{D}_m^{\preceq} G_0 \rightarrow \mathcal{D}_m \mathbf{p}$ ,  $K_1 \mapsto \iota^* K_1$ ,  $c = -2a - \rho$  (see [L12, 8.3(a)]),  $K$  replaced by  $b_!(K) * K' \in \mathcal{D}_m(G_{1,0})$  and  $c' = 2a + \rho$  we see that we have canonically

$$(\iota^*(b_!(K) * K'))^{\{-2a-\rho\}} \subset (\iota^*(b_!(K) * K'))^{\{0\}}.$$

In particular, if  $L, L' \in \mathcal{C}_0^c \mathcal{B}^2$  then we have canonically

$$(\iota^*(\underline{\chi}_{\epsilon,s}(L') * \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{-2a-\rho\}} \subset (\iota^*(\underline{\chi}_{\epsilon,s}(L') * \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{0\}}.$$

Using the equality

$$(\iota^*(\underline{\chi}_{\epsilon,s}(L') * \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{-2a-\rho\}} = (\iota^*(\underline{\chi}_{1,0}(L \bullet \underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L')))))^{\{-2a-\rho\}}$$

which comes from 5.1, we deduce that we have canonically

$$(\iota^*(\underline{\chi}_{1,0}(L \bullet \underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L')))))^{\{-2a-\rho\}} \subset (\iota^*(\underline{\chi}_{\epsilon,s}(L') * \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{0\}}$$

or equivalently, using (a) with  $K, K'$  replaced by  $b^* \underline{\chi}_{\epsilon,s}(L')$ ,  $\underline{\chi}_{\epsilon^{-1},-s}(L)$ :

$$\begin{aligned} (\iota^*(\underline{\chi}_{1,0}(L \bullet \underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L')))))^{\{-2a-\rho\}} &\subset \text{Hom}_{\mathcal{C}^c G_{\epsilon^{-1},-s}}(\mathfrak{D}(b^* \underline{\chi}_{\epsilon,s}(L')), \underline{\chi}_{\epsilon^{-1},-s}(L)) \\ &= \text{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\mathfrak{D}(b! \underline{\chi}_{\epsilon^{-1},-s}(L)), \underline{\chi}_{\epsilon,s}(L')). \end{aligned}$$

Using now [L12, 8.6(c)], we deduce that we have canonically

$$\text{Hom}_{\mathcal{C}^c \mathcal{B}^2}(\mathbf{1}', L \bullet \underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s} L') \subset \text{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\mathfrak{D}(b! \underline{\chi}_{\epsilon^{-1},-s}(L)), \underline{\chi}_{\epsilon,s}(L'))$$

where  $\mathbf{1}'$  is as in [L12, 8.6] or equivalently (see [L12, 8.7]):

$$\text{Hom}_{\mathcal{C}^c \mathcal{B}^2}(\mathfrak{D}(b! L), \underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s} L') \subset \text{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\mathfrak{D}(b! \underline{\chi}_{\epsilon^{-1},-s}(L)), \underline{\chi}_{\epsilon,s}(L'))$$

where  $b' : \mathcal{B}^2 \rightarrow \mathcal{B}^2$  is  $(B, B') \mapsto (B', B)$ . We now set  ${}^1L = \mathfrak{D}(b! L)$  and note that

$$\mathfrak{D}(b! \underline{\chi}_{\epsilon^{-1},-s}(L)) = \mathfrak{D}(\underline{\chi}_{\epsilon,s}(b! L)) = \underline{\chi}_{\epsilon,s}(\mathfrak{D}(b! L)) = \underline{\chi}_{\epsilon,s}({}^1L),$$

see 1.3, 1.10(a). We obtain

$$(b) \quad \text{Hom}_{\mathcal{C}^c \mathcal{B}^2}({}^1L, \underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s} L') \subset \text{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}({}^1L), \underline{\chi}_{\epsilon,s}(L'))$$

for any  ${}^1L, L' \in \mathcal{C}_0^c \mathcal{B}^2$ .

We have the following result which is a weak form of an adjunction formula, of which the full form will be proved in 6.6.

**Proposition 5.3.** *Let  $s \in \mathbf{Z}$ . For any  ${}^1L, L' \in \mathcal{C}_0^c \mathcal{B}^2$  we have canonically*

$$(a) \quad \text{Hom}_{\mathcal{C}^c \mathcal{B}^2}({}^1L, \underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s}(L')) = \text{Hom}_{\mathcal{C}^c G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}({}^1L), \underline{\chi}_{\epsilon,s}(L'))$$

We can assume that  ${}^1L = \mathbf{L}_z, L' = \mathbf{L}_u$  where  $z, u \in \mathbf{c}$ . By 4.9(a) and 1.8(b), both sides of the inclusion 5.2(b) have dimension  $\sum_{y \in \mathbf{c}} \tau(t_{y^{-1}} t_z t_{\epsilon(y)} t_{u^{-1}})$ . Hence that inclusion is an equality. The proposition is proved. (The case where  $\epsilon = 1, s = 0$  is treated in [L12, 8.9].)

6. EQUIVALENCE OF  $\mathcal{C}^c G_{\epsilon,s}$  WITH THE  $\epsilon$ -CENTRE OF  $\mathcal{C}^c \mathcal{B}^2$ 

**6.1.** For  $\epsilon' \in \mathfrak{A}$  such that  $\epsilon'(\mathbf{c}) = \mathbf{c}$  and  $s, s' \in \mathbf{Z}$ , the bifunctor  $\mathcal{C}_0^c G_{\epsilon,s} \times \mathcal{C}_0^c G_{\epsilon',s'} \rightarrow \mathcal{C}_0^c G_{\epsilon\epsilon',s+s'}$ ,  $K, K' \mapsto K \underline{*} K'$  in 3.6 defines a bifunctor  $\mathcal{C}^c G_{\epsilon,s} \times \mathcal{C}^c G_{\epsilon',s'} \rightarrow \mathcal{C}^c G_{\epsilon\epsilon',s+s'}$  denoted again by  $K, K' \mapsto K \underline{*} K'$  as follows. Let  $K \in \mathcal{C}^c G_{\epsilon,s}$ ,  $K' \in \mathcal{C}^c G_{\epsilon',s'}$ ; we choose mixed structures of pure weight 0 on  $K, K'$  (this is possible if  $s_0$  in 0.3 is large enough), we define  $K \underline{*} K' \in \mathcal{C}_0^c G_{\epsilon\epsilon',s+s'}$  as in 3.6 in terms of these mixed structures and we then disregard the mixed structure on  $K \underline{*} K'$ . The resulting object of  $\mathcal{C}^c G_{\epsilon\epsilon',s+s'}$  is denoted again by  $K \underline{*} K'$ ; it is independent of the choices made.

In the same way, the bifunctor  $\mathcal{C}_0^c \mathcal{B}^2 \times \mathcal{C}_0^c \mathcal{B}^2 \rightarrow \mathcal{C}_0^c \mathcal{B}^2$ ,  $L, L' \mapsto L \bullet L'$  gives rise to a bifunctor  $\mathcal{C}^c \mathcal{B}^2 \times \mathcal{C}^c \mathcal{B}^2 \rightarrow \mathcal{C}^c \mathcal{B}^2$  denoted again by  $L, L' \mapsto L \bullet L'$ ; the functor  $\underline{\chi}_{\epsilon,s} : \mathcal{C}_0^c \mathcal{B}^2 \rightarrow \mathcal{C}_0^c G_{\epsilon,s}$  gives rise to a functor  $\mathcal{C}^c \mathcal{B}^2 \rightarrow \mathcal{C}^c G_{\epsilon,s}$  denoted again by  $\underline{\chi}_{\epsilon,s}$  (it is again called *truncated induction*); the functor  $\underline{\zeta}_{\epsilon,s} : \mathcal{C}_0^c G_{\epsilon,s} \rightarrow \mathcal{C}_0^c \mathcal{B}^2$  gives rise to a functor  $\mathcal{C}^c G_{\epsilon,s} \rightarrow \mathcal{C}^c \mathcal{B}^2$  denoted again by  $\underline{\zeta}_{\epsilon,s}$  (it is again called *truncated restriction*).

The operation  $K \underline{*} K'$  is again called *truncated convolution*. It has a canonical associativity isomorphism (deduced from that in 3.7) which satisfies the pentagon property.

The operation  $L \bullet L'$  makes  $\mathcal{C}^c \mathcal{B}^2$  into a monoidal abelian category (see also [L9]) which has a unit object (see [L12, 9.2]) and is rigid (see [L12, 9.3]).

Note that  $L \mapsto {}^\epsilon L$  (see 1.6) can be regarded as a functor  $\mathcal{C}^c \mathcal{B}^2 \rightarrow \mathcal{C}^c \mathcal{B}^2$ .

**6.2.** Extending slightly a definition in [Mu, 3.1] we define a  $\epsilon$ -half braiding for an object  $\mathcal{L} \in \mathcal{C}^c \mathcal{B}^2$  as a collection  $e_{\mathcal{L}} = \{e_{\mathcal{L}}(L); L \in \mathcal{C}^c \mathcal{B}^2\}$  where  $e_{\mathcal{L}}(L)$  are isomorphisms  ${}^\epsilon L \bullet \mathcal{L} \xrightarrow{\sim} \mathcal{L} \bullet L$  such that (i),(ii) below hold:

(i) If  $L \xrightarrow{t} L'$  is any morphism in  $\mathcal{C}^c \mathcal{B}^2$  then the diagram

$$\begin{array}{ccc} {}^\epsilon L \bullet \mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L} \bullet L \\ {}^\epsilon t \bullet 1 \downarrow & & 1 \bullet t \downarrow \\ {}^\epsilon L' \bullet \mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L')} & \mathcal{L} \bullet L' \end{array}$$

is commutative.

(ii) If  $L, L' \in \mathcal{C}^c \mathcal{B}^2$  then  $e_{\mathcal{L}}(L \bullet L') : {}^\epsilon(L \bullet L') \bullet \mathcal{L} \rightarrow \mathcal{L} \bullet (L \bullet L')$  is equal to the composition

$${}^\epsilon L \bullet {}^\epsilon L' \bullet \mathcal{L} \xrightarrow{1 \bullet e_{\mathcal{L}}(L')} {}^\epsilon L \bullet \mathcal{L} \bullet L' \xrightarrow{e_{\mathcal{L}}(L) \bullet 1} \mathcal{L} \bullet L \bullet L'.$$

When  $\epsilon = 1$ , this reduces to the definition of a half-braiding for  $\mathcal{L}$  given in [Mu, 3.1]. Let  $\mathcal{Z}_{\epsilon}^c$  the category whose objects are the pairs consisting of an object  $\mathcal{L}$  of  $\mathcal{C}^c \mathcal{B}^2$  and an  $\epsilon$ -half braiding for  $\mathcal{L}$ . For  $(\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'})$  in  $\mathcal{Z}_{\epsilon}^c$  we define

$\text{Hom}_{\mathcal{Z}_\epsilon^\mathbf{c}}((\mathcal{L}, e_\mathcal{L}), (\mathcal{L}', e_{\mathcal{L}'}))$  to be the vector space consisting of all  $t \in \text{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(\mathcal{L}, \mathcal{L}')$  such that for any  $L \in \mathcal{C}^\mathbf{c}\mathcal{B}^2$  the diagram

$$\begin{array}{ccc} {}^\epsilon L \bullet \mathcal{L} & \xrightarrow{e_\mathcal{L}(L)} & \mathcal{L} \bullet L \\ 1 \bullet t \downarrow & & t \bullet 1 \downarrow \\ {}^\epsilon L \bullet \mathcal{L}' & \xrightarrow{e_{\mathcal{L}'}(L)} & \mathcal{L}' \bullet L \end{array}$$

is commutative. We say that  $\mathcal{Z}_\epsilon^\mathbf{c}$  is the  $\epsilon$ -centre of  $\mathcal{C}^\mathbf{c}\mathcal{B}^2$ . (When  $\epsilon = 1$ , it reduces to the centre of  $\mathcal{C}^\mathbf{c}\mathcal{B}^2$ , see [Mu, 3.2].)

If  $s \in \mathbf{Z}$  and  $K \in \mathcal{C}^\mathbf{c}G_s$  then the isomorphisms 2.9(a) provide an  $\epsilon$ -half braiding for  $\zeta_{\epsilon,s}(K) \in \mathcal{C}^\mathbf{c}\mathcal{B}^2$  so that  $\zeta_{\epsilon,s}(K)$  can be naturally viewed as an object of  $\mathcal{Z}_\epsilon^\mathbf{c}$  denoted by  $\overline{\zeta_{\epsilon,s}(K)}$ . (Note that 2.9 is stated in the mixed category but, as above, it implies the corresponding result in the unmixed category.) Then  $K \mapsto \overline{\zeta_{\epsilon,s}(K)}$  is a functor  $\mathcal{C}^\mathbf{c}G_{\epsilon,s} \rightarrow \mathcal{Z}_\epsilon^\mathbf{c}$ . We have the following result.

**Theorem 6.3.** *Let  $s \in \mathbf{Z}$ . The functor  $\mathcal{C}^\mathbf{c}G_{\epsilon,s} \rightarrow \mathcal{Z}_\epsilon^\mathbf{c}$ ,  $K \mapsto \overline{\zeta_{\epsilon,s}(K)}$  is an equivalence of categories.*

When  $\epsilon = 1, s = 0$  this reduces to [L12, 9.5]. The general case will be proved in 6.5.

Note that, when combined with 1.6(b), the theorem yields for any  $F \in G_{\epsilon,s}$  (with  $s > 0$ ) a category equivalence

$$(a) \quad \text{Rep}^\mathbf{c}(G^F) \xrightarrow{\sim} \mathcal{Z}_\epsilon^\mathbf{c}.$$

**6.4.** By a variation of a general result on semisimple rigid monoidal categories in [ENO, Proposition 5.4], for any  $L \in \mathcal{C}^\mathbf{c}\mathcal{B}^2$  one can define directly an  $\epsilon$ -half braiding on the object  $I_\epsilon(L) := \bigoplus_{y \in \mathbf{c}} \mathbf{L}_y \bullet L \bullet \mathbf{L}_{\epsilon(y)^{-1}}$  of  $\mathcal{C}^\mathbf{c}\mathcal{B}^2$  such that, denoting by  $\overline{I_\epsilon(L)}$  the corresponding object of  $\mathcal{Z}_\epsilon^\mathbf{c}$ , we have canonically

$$(a) \quad \text{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(L, \mathcal{L}) = \text{Hom}_{\mathcal{Z}_\epsilon^\mathbf{c}}(\overline{I_\epsilon(L)}, (\mathcal{L}, e_\mathcal{L}))$$

for any  $(\mathcal{L}, e_\mathcal{L}) \in \mathcal{Z}_\epsilon^\mathbf{c}$ .

The  $\epsilon$ -half braiding on  $I_\epsilon(L)$  can be described as follows: for any  $L' \in \mathcal{C}^\mathbf{c}\mathcal{B}^2$  we have canonically

$$\begin{aligned} {}^\epsilon L' \bullet I_\epsilon(L) &= \bigoplus_{y \in \mathbf{c}} {}^\epsilon L' \bullet \mathbf{L}_y \bullet L \bullet \mathbf{L}_{\epsilon(y)^{-1}} = \bigoplus_{y, z \in \mathbf{c}} \text{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(\mathbf{L}_z, {}^\epsilon L' \bullet \mathbf{L}_y) \otimes \mathbf{L}_z \bullet L \bullet \mathbf{L}_{\epsilon(y)^{-1}} \\ &= \bigoplus_{y, z \in \mathbf{c}} \text{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(\mathbf{L}_{y^{-1}}, \mathbf{L}_{z^{-1}} \bullet {}^\epsilon L') \otimes \mathbf{L}_z \bullet L \bullet \mathbf{L}_{\epsilon(y)^{-1}} \\ &= \bigoplus_{y, z \in \mathbf{c}} \text{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(\mathbf{L}_{\epsilon(y)^{-1}}, \mathbf{L}_{\epsilon(z)^{-1}} \bullet L') \otimes \mathbf{L}_z \bullet L \bullet \mathbf{L}_{\epsilon(y)^{-1}} \\ &= \bigoplus_{z \in \mathbf{c}} \mathbf{L}_z \bullet L \bullet \mathbf{L}_{\epsilon(z)^{-1}} \bullet L' = I_\epsilon(L) \bullet L'. \end{aligned}$$

By a variation of results in [Mu, 3.3], [ENO, 2.15], we see that  $\mathcal{Z}_\epsilon^\mathbf{c}$  is a semisimple  $\bar{\mathbf{Q}}_l$ -linear category with finitely many simple objects up to isomorphism. Note that

(b) *if  $\sigma = (\mathcal{L}, e_\mathcal{L})$  is a simple object of  $\mathcal{Z}_\epsilon^\mathbf{c}$  then  $\sigma$  is a summand of  $\overline{I_\epsilon(\mathbf{L}_z)}$  for some  $z \in \mathbf{c}$ .*

Indeed, let  $z \in \mathbf{c}$  be such that  $\mathbf{L}_z$  is a summand of  $\mathcal{L}$  in  $\mathcal{C}^\mathbf{c}\mathcal{B}^2$ ; then by (a),  $\sigma$  is a summand of  $\overline{I_\epsilon(\mathbf{L}_z)}$ .

**6.5.** Let  $s \in \mathbf{Z}$ . For  $x \in \mathbf{c}$  we have canonically  $\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x = I_\epsilon(\mathbf{L}_x)$  as objects of  $\mathcal{C}^\mathbf{c}\mathcal{B}^2$ , see Theorem 4.8. This identification is compatible with the  $\epsilon$ -half braidings (see 6.2, 6.4). (When  $\epsilon = 1, s = 0$  this follows from the last commutative diagram in [L12, 7.9]; in the general case we have an analogous commutative diagram, which is established using the results in Section 4.) It follows that

$$(a) \quad \overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x} = \overline{I_\epsilon(\mathbf{L}_x)}.$$

Using this and 6.4(a) with  $\mathcal{L} = \overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\tilde{L}}$ ,  $\tilde{L} \in \mathcal{C}^\mathbf{c}\mathcal{B}^2$ , we see that

$$\mathrm{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(\mathbf{L}_x, \underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\tilde{L}) = \mathrm{Hom}_{\mathcal{Z}_\epsilon^\mathbf{c}}(\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x}, \overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\tilde{L}}).$$

Combining this with 5.3 we obtain for  $\tilde{L} = \mathbf{L}_{x'}$  (with  $x' \in \mathbf{c}$ ):

$$(b) \quad \mathbf{A}_{x,x'} = \mathbf{A}'_{x,x'}$$

where

$$\mathbf{A}_{x,x'} = \mathrm{Hom}_{\mathcal{C}^\mathbf{c}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_x), \underline{\chi}_{\epsilon,s}(\mathbf{L}_{x'})), \mathbf{A}'_{x,x'} = \mathrm{Hom}_{\mathcal{Z}_\epsilon^\mathbf{c}}(\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x}, \overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_{x'}}).$$

Note that the identification (b) is induced by the functor  $K \mapsto \overline{\underline{\zeta}_{\epsilon,s}(K)}$ . Let  $\mathbf{A} = \bigoplus_{x,x' \in \mathbf{c}} \mathbf{A}_{x,x'}$ ,  $\mathbf{A}' = \bigoplus_{x,x' \in \mathbf{c}} \mathbf{A}'_{x,x'}$ . Then from (b) we have  $\mathbf{A} = \mathbf{A}'$ . Note that this identification is compatible with the obvious algebra structures of  $\mathbf{A}, \mathbf{A}'$ .

For any  $A \in CS_{\epsilon,s,\mathbf{c}}$  we denote by  $\mathbf{A}_A$  the set of all  $f \in \mathbf{A}$  such that for any  $x, x'$ , the  $(x, x')$ -component of  $f$  maps the  $A$ -isotypic component of  $\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)$  to the  $A$ -isotypic component of  $\underline{\chi}_{\epsilon,s}(\mathbf{L}_{x'})$  and any other isotypic component of  $\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)$  to 0. Then  $\mathbf{A} = \bigoplus_{A \in CS_{\epsilon,s,\mathbf{c}}} \mathbf{A}_A$  is the decomposition of  $\mathbf{A}$  into a sum of simple algebras (each  $\mathbf{A}_A$  is  $\neq 0$  since, by 1.5(b) and 1.8(a), any  $A$  is a summand of some  $\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)$ ).

Let  $\mathfrak{S}$  be a set of representatives for the isomorphism classes of simple objects of  $\mathcal{Z}_\epsilon^\mathbf{c}$ . For any  $\sigma \in \mathfrak{S}$  we denote by  $\mathbf{A}'_\sigma$  the set of all  $f' \in \mathbf{A}'$  such that for any  $x, x'$ , the  $(x, x')$ -component of  $f'$  maps the  $\sigma$ -isotypic component of  $\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)}$  to the  $\sigma$ -isotypic component of  $\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(\mathbf{L}_{x'})}$  and any other isotypic component of

$\overline{\zeta_{\epsilon,s}\chi_{\epsilon,s}}(\mathbf{L}_x)$  to 0. Then  $\mathbf{A}' = \bigoplus_{\sigma \in \mathfrak{S}} \mathbf{A}'_{\sigma}$  is the decomposition of  $\mathbf{A}'$  into a sum of simple algebras (each  $\mathbf{A}'_{\sigma}$  is  $\neq 0$  since any  $\sigma$  is a summand of some  $\overline{\zeta_{\epsilon,s}\chi_{\epsilon,s}}(\mathbf{L}_z)$  (we use 6.4(b), 6.5(a)).

Since  $\mathbf{A} = \mathbf{A}'$ , from the uniqueness of decomposition of a semisimple algebra as a direct sum of simple algebras, we see that there is a unique bijection  $CS_{\epsilon,s,\mathbf{c}} \leftrightarrow \mathfrak{S}$ ,  $A \leftrightarrow \sigma_A$  such that  $\mathbf{A}_A = \mathbf{A}'_{\sigma_A}$  for any  $A \in CS_{\epsilon,s,\mathbf{c}}$ . From the definitions we now see that for any  $A \in CS_{\epsilon,s,\mathbf{c}}$  we have  $\overline{\zeta_{\epsilon,s}}A \cong \sigma_A$ . Therefore Theorem 6.3 holds.

**Theorem 6.6.** *Let  $s \in \mathbf{Z}$ . Let  $L \in \mathcal{C}^c\mathcal{B}^2$ ,  $K \in \mathcal{C}^cG_{\epsilon,s}$ . We have canonically*

$$(a) \quad \text{Hom}_{\mathcal{C}^c\mathcal{B}^2}(L, \zeta_{\epsilon,s}(K)) = \text{Hom}_{\mathcal{C}^cG_{\epsilon,s}}(\chi_{\epsilon,s}(L), K).$$

Moreover, in  $\mathcal{C}^c\mathcal{B}^2$  we have  $\zeta_{\epsilon,s}(K) \cong \bigoplus_{z \in \mathbf{c}^0} \mathbf{L}_z^{\oplus m_z}$  where  $\mathbf{c}^0$  is as in 1.5 and  $m_z \in \mathbf{N}$ .

From 6.3, 6.5, we see that

$$\begin{aligned} \text{Hom}_{\mathcal{C}^cG_{\epsilon,s}}(\chi_{\epsilon,s}(L), K) &= \text{Hom}_{\mathcal{Z}_{\epsilon}^c}(\overline{\zeta_{\epsilon,s}\chi_{\epsilon,s}}(L), \overline{\zeta_{\epsilon,s}}K) \\ &= \text{Hom}_{\mathcal{Z}_{\epsilon}^c}(\overline{I_{\epsilon}(L)}, \overline{\zeta_{\epsilon,s}}K). \end{aligned}$$

Using 6.4(a) we see that

$$\text{Hom}_{\mathcal{Z}_{\epsilon}^c}(\overline{I_{\epsilon}(L)}, \overline{\zeta_{\epsilon,s}}K) = \text{Hom}_{\mathcal{C}^c\mathcal{B}^2}(L, \zeta_{\epsilon,s}(K))$$

and (a) follows. To prove the second assertion of the theorem it is enough to show that for any  $z \in \mathbf{c} - \mathbf{c}^0$  we have  $\text{Hom}_{\mathcal{C}^c\mathcal{B}^2}(\mathbf{L}_z, \zeta_{\epsilon,s}(K)) = 0$ ; by (a), it is enough to show that  $\chi_{\epsilon,s}(\mathbf{L}_z) = 0$  and this follows from 1.5(c). (The case where  $\epsilon = 1, s = 0$  is just [L12, 9.8].)

**6.7.** Let  $s \in \mathbf{Z}$ . For  $K \in \mathcal{C}^cG_{\epsilon,s}$  we have canonically

$$(a) \quad \mathfrak{D}(\zeta_{\epsilon,s}(\mathfrak{D}(K))) = \zeta_{\epsilon,s}(K).$$

When  $\epsilon = 1, s = 0$  this is proved in [L12, 9.9]. The proof in the general case is entirely similar; it uses 6.6(a), 1.10(a).

**6.8.** In this subsection we assume that  $\epsilon = 1$ . The monoidal structure on  $\mathcal{C}^c\mathcal{B}^2$  induces a monoidal structure on  $\mathcal{Z}_1^c$ . Moreover, the category

$$(a) \quad \sqcup_{s \in \mathbf{Z}} \mathcal{C}^cG_{1,s} = \mathcal{C}^cG_{1,0} \sqcup_{s \in \mathbf{Z}; s \neq 0} \text{Rep}^c(G^{F_0^s})$$

(see 1.6(b)) has a monoidal structure given by truncated convolution, see 6.1. Moreover, 6.3 provides a functor from (a) to  $\mathcal{Z}_1^c$  which is an equivalence when restricted to any  $\mathcal{C}^cG_s$ . This functor is compatible with the monoidal structures (this can be deduced from 3.8 and from the fact that the monoidal structure of  $\mathcal{Z}_1^c$  is equivalent to its opposite). Note that  $\mathcal{C}^cG_{1,0}$  is a monoidal subcategory of (a), whose unit object, described in [L12, 9.10], is also a unit object for the monoidal category (a).



**6.9.** The functor  $L \mapsto {}^\epsilon L$  from  $\mathcal{C}^c \mathcal{B}^2$  into itself induces a functor  $\mathcal{Z}_\epsilon^c \rightarrow \mathcal{Z}_\epsilon^c$  which carries any simple object  $(L, e_L)$  of  $\mathcal{Z}_\epsilon^c$  into an object isomorphic to  $(L, e_L)$ ; this follows from 2.10(c), using Theorem 6.3.

**6.10.** Let  $s \in \mathbf{Z}$ . For any  $A \in CS_{\epsilon, s, \mathbf{c}}$  and any  $x \in \mathbf{c}$  we denote by  $n_{A, x}$  the multiplicity of  $A$  in  $\chi_{\epsilon, s} \mathbf{L}_x \in \mathcal{C}^c G_{\epsilon, s}$ . From Theorem 6.3 and its proof we see that if  $\sigma$  is the simple object of  $\mathcal{Z}_\epsilon^c$  corresponding to  $A$ , then  $n_{A, x}$  is equal to the multiplicity of  $\sigma$  in  $\overline{I_\epsilon(\mathbf{L}_x)} \in \mathcal{Z}_\epsilon^c$ . In particular,  $n_{A, x}$  is independent of  $s$ .

## 7. RELATION WITH SOERGEL BIMODULES

**7.1.** Let  $R$  be the algebra of polynomials functions on a fixed reflection representation of  $W$  (over  $\mathbf{Q}_l$ ). Then for each  $x \in W$ , the indecomposable Soergel graded  $R$ -bimodule  $B_x$  is defined as in [So, 6.16]. Let  $C_\mathbf{c}$  be the category of graded  $R$ -bimodules which are isomorphic to finite direct sums of graded  $R$ -bimodules of the form  $B_x$  ( $x \in \mathbf{c}$ ) without shift. There is a well defined functor  $M \mapsto {}^\epsilon M$  from  $C_\mathbf{c}$  to  $C_\mathbf{c}$  which is linear and satisfies  ${}^\epsilon B_x = B_{\epsilon^{-1}(x)}$  for  $x \in \mathbf{c}$ . Now  $C_\mathbf{c}$  has a natural monoidal structure (see [L12, 10.1] defined purely in terms of  $R, W, \mathbf{c}$ . (Its definition makes use of the results in [EW].) From the definition we see that  $C_\mathbf{c}$  is equivalent to  $\mathcal{C}^c \mathcal{B}^2$  as monoidal categories so that  $M \mapsto {}^\epsilon M$  corresponds to  $L \mapsto {}^\epsilon L$  from  $\mathcal{C}^c \mathcal{B}^2$  to itself. Then the  $\epsilon$ -centre of  $C_\mathbf{c}$  is defined as in 6.2. It is naturally equivalent to  $\mathcal{Z}_\epsilon^c$ . Thus we can restate Theorem 6.3 as follows.

(a) *For any  $s \in \mathbf{Z}$ , the category  $\mathcal{C}^c G_{\epsilon, s} \rightarrow \mathcal{Z}_\epsilon^c$  is naturally equivalent to the  $\epsilon$ -centre of the monoidal category  $C_\mathbf{c}$ .*

This, combined with 1.6(b), shows that for  $F \in G_{\epsilon, s}$  (with  $s > 0$ ), the category  $\text{Rep}^c(G^F)$  is equivalent to the  $\epsilon$ -centre of the monoidal category  $C_\mathbf{c}$ ; thus, the set of simple objects of  $\text{Rep}^c(G^F)$  is not only independent of  $s$  but also independent of the characteristic of  $\mathbf{k}$ , since the  $\epsilon$ -centre of  $C_\mathbf{c}$  is so. (Here we identify  $\mathbf{Q}_l$  with the complex numbers.)

**7.2.** As mentioned in [L12, 10.1], the definition of the monoidal category  $C_\mathbf{c}$  makes sense even when  $W$  is replaced by any (say finite, irreducible) Coxeter group and  $\mathbf{c}$  is a two-sided cell in  $W$ . Assume now that  $\epsilon : W \rightarrow W$  is an automorphism of  $W$  which leaves stable the set of simple reflections and leaves stable  $\mathbf{c}$ . Then the definition of the  $\epsilon$ -centre of  $C_\mathbf{c}$  makes sense even if  $W$  is noncrystallographic. We expect that the indecomposable objects of the  $\epsilon$ -centre of  $C_\mathbf{c}$  are in bijection with the “unipotent characters” associated to  $W, \epsilon, \mathbf{c}$  in [L5]. (For  $\epsilon = 1$  this expectation has already been stated in [L12, 10.1].)

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